Non-archimedean geometry - 1
Introduction

§-1 : Complex geometry

Let X be a variety over $\mathbb{C}$.

Then $X(\mathbb{C})$ is a top. space with a Zariski topology. But $\mathbb{C}$ has a Euclidean topology, we can endow $X(\mathbb{C})$ with this topology!

- If $X = \mathbb{A}^n_{\mathbb{C}} \rightsquigarrow X(\mathbb{C}) = \mathbb{C}^n$ (product topology)
- If $X = V(I) \subset \mathbb{A}^n_{\mathbb{C}} \rightsquigarrow V(I)(\mathbb{C}) \subset \mathbb{C}^n$ (subspace topology)
- More generally glue...

We get a functor

$$\mathrm{An}: \left\{\begin{matrix}\text{finite type}\\ \text{schemes}/\mathbb{C}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{complex analytic}\\ \text{spaces}\end{matrix}\right\}$$

$$\begin{matrix}\cup & & \cup\end{matrix}$$

$$\left\{\begin{matrix}\text{smooth varieties}\\ \text{over } \mathbb{C}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{complex}\\ \text{manifolds}\end{matrix}\right\}$$

Serre proved GAGA theorem:

- $\mathrm{An}: \left\{\begin{matrix}\text{coherent sheaves}\\ \text{over } X\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{coherent sheaves}\\ \text{over } X(\mathbb{C})\end{matrix}\right\}$

  for $\mathcal{O}_{X(\mathbb{C}),\mathrm{hol}}$

- Equiv. between proj. alg. curves and compact Riemann surfaces

- Chow's theorem: $Z \subset \mathbb{P}^n(\mathbb{C})$ closed, then Z is a subvariety!

Why is this helpful?

1. Hodge theory: Let Y be a compact Kähler manifold. Then we have a decomp.:

$$H^n(Y,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(Y) \quad \text{and} \quad H^{p,q} \cong \overline{H^{q,p}}$$

where $H^{p,q}(Y)$ is the Dolbeault cohomology

$$H^{p,q}(Y) \cong H^p(Y, \Omega^q_Y)$$

(Proj. varieties are Kähler)

Hodge symmetry is purely transcendental!

2. Elliptic curves

$$\left\{ \begin{matrix} \text{Lattices} \\ \Lambda \subset \mathbb{C} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{Elliptic curves} \\ \text{over } \mathbb{C} \end{matrix} \right\}$$

via Weierstraß-functions

$E = \mathbb{C}/\Lambda \rightsquigarrow E[N] = \frac{1}{N}\Lambda/\Lambda$

Basic principle:

Topology of $\mathbb{C}$ endow $X/\mathbb{C}$ variety with more structure, leading to a better understanding not only of $X(\mathbb{C})$, but also $X/\mathbb{C}$.

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§0 A first attempt: p-adic manifolds

Question: $\mathbb{Q}_p$ or $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$ or $\mathbb{F}_p((t))$ also have a non-trivial topology. Can we build manifolds out of these?

One can try:

Def.: A (locally analytic) manifold $(M, \mathcal{A})$ is a Hausdorff top. space M equipped with a maxi. Atlas, where the transition maps are required to be locally analytic.

(= locally a power series)

$$K = \mathbb{Q}_p \qquad e_i : \mathcal{O}_K^n \to U_i \subset M$$

$$e_i : K^n \to U_i \subset M$$

$$\varphi_{ij} : e_i^{-1}(U_i \cap U_j) \to U_i \cap U_j \leftarrow e_j^{-1}(U_i \cap U_j)$$

Problem: The topology on $\mathbb{Z}_p$ or $\mathbb{Q}_p$ is not intuitive! So for example we get phenomena like:

$$\mathbb{Z}_p = \{|x| < 1\} \amalg \{|x| = 1\}$$

clopen

Theorem: - Any compact p-adic manifold is isomorphic to a disjoint union of r p-adic balls of dimension n for $r \geq 1$

- r is unique mod $(p-1)$

Still has application e.g. p-adic Lie groups.

But we need something else.

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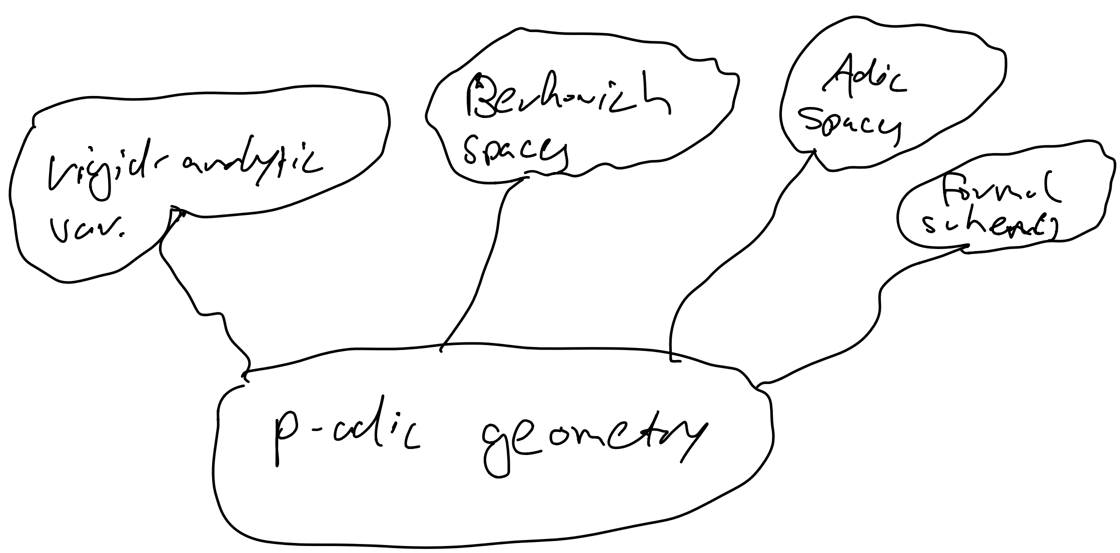
§1 The p-adic jungle



- Rigid-analytic varieties

Very closely related to our intuition

E.g. $V(f) \subset \mathbb{B}^n(\bar{K})$

where $f \notin k[T]$, but

$f \in k\langle T\rangle$ = conv. powerseries

↳ Get uniformization of elliptic curves

"$E \cong K^*/q^{\mathbb{Z}}$" $\quad (\mathbb{C}/\Lambda \xrightarrow{\exp} \mathbb{C}^*/q^{\mathbb{Z}})$

↳ Have to use a Grothendieck-topology via admissible opens

- Berkovich spaces: Fix this by adding more points to your top. space. We end up with something geometric. These spaces, in large generality, Hausdorff..

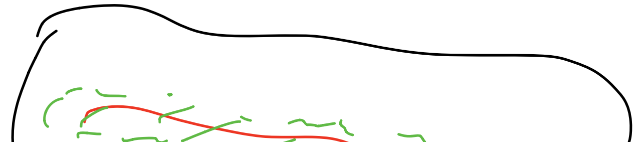
Add more points, but not prime ideals, but semi-norms $|\cdot|: T \to \mathbb{R}_{\geq 0}$

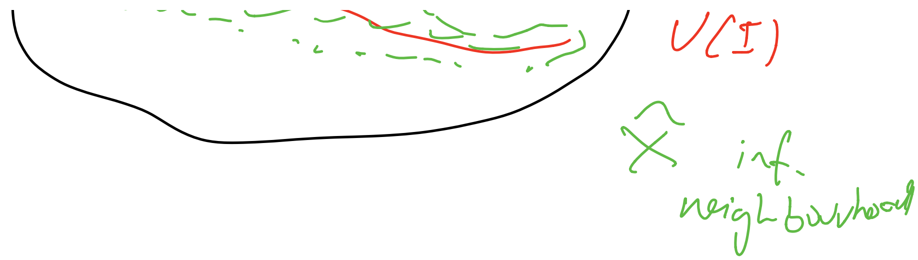
Use this for e.g. mirror symmetry, but also for compactifications of Bruhat-Tits buildings.

- Adic spaces: Similar to Berkovich spaces, but more of arithmetic nature. Can be stated in a rather general set-up. Important application: Perfectoid spaces

- Formal schemes: They are connected to all of the above via "formal models". Very closely related to algebraic geometry.

X scheme





Abstract question:

Ring ~ Comm. algebra $\Rightarrow$ Algebraic geometry

Rings + topology $\Rightarrow$???

$$X \cong \varprojlim(\mathfrak{X})$$

formal models of X