G = top. group

G = top. space

= group

$\cdot : G \times G \to G$ cont.

$^{-1} : G \to G$ — " —

$G_1 \xrightarrow{\varphi} G_2$ φ = cont + hom.

• G = top. group

equivalent: $e = 1$ = closed

$\Updownarrow$

G = Hausdorff

$\Downarrow$: $xy^{-1}: G \times G \to G$

(x,y) $x \neq y$ diagonal $\leftarrow 1$

$\square$ $(x,y) \in U \times V$, $U, V \subseteq G$

$U \cap V = \emptyset$

• top group = uniform str.

$U \ni 1$ $xy^{-1} \in U$

adic and f adic

$+, \cdot, -$

• non-archimedean ring: top. ring where 0 has a basis of open neigh. which are subgroups

• adic rings: A = top. ring s.t. $\forall$ $I \trianglelefteq A$ s.t. $(I^n)_{n=1}^{\infty}$ subbasis of open neigh. of A

• f adic rings: A = top ring s.t. $\exists$ open subring $A_0 \subset A$ s.t. A_0 with subspace top. is adic and $\exists$ $I \trianglelefteq A_0$ gives the top. and I = fin. gen.

A_0 = ring of def.

(A_0, I) = pair of def. Huber

Ex: $\mathbb{Z}_p = \mathbb{Z}_p$, $I = (p) \trianglelefteq \mathbb{Z}_p$

$\Leftarrow$ adic ring.

$\mathbb{Q}_p \supset \mathbb{Z}_p$ f-adic ring

$(\mathbb{Z}_p, (p))$ = pair of def.

$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p} \supset \mathcal{O} = \overline{\mathbb{Z}_p}$

f adic ring, $(\mathcal{O}, (p))$

not Noetherian

Prop: A = top. ring. Equivalent:

1.) A = f adic ring

2.) $\exists$ an open subgroup $U \subset A$ s.t. the top. is gen. by U^n + $\exists$ fin. subset $T \subset U$

def. A = top. ring s.t. $T \cdot U = U^2 \subseteq U$

$E \subset A$. We say that E is bounded if $\forall$ open $U \ni 0$ $\exists$ open $V \ni 0$

$V \cdot e \subset U$ $\forall e \in E$ $v \in V$.

G = top. ring.

this def $\iff$ usual defn.

A = top. ring $U, V \subseteq A$

$U \cdot V = \{$ fin. sums of $u \cdot v$ | $u \in U, v \in V \}$

$U_1 \cdot U_2 \cdots U_n$, $U^n = \underbrace{U \cdots U}_{n \text{ times}}$

$U^n \subseteq U^{n-1} \subset U$

$U^2 \subset U$ ~~U^n open?~~

U

$T(n) = \{ t_1 \cdot t_2 \cdots t_n \mid t_i \in T \}$

$T \subseteq A$

Lemma: A = top. ring, $A_0 \subset$ subring. Then the following are equivalent:

1.) A = f adic, A_0 = ring of def.

2.) A = f adic, A_0 = open $\dots$

3.) A = satisfies the second def. of f adic + A_0 = open and bounded $\exists$ $I \trianglelefteq A_0$

1.) $\Rightarrow$ 2.) I^n = basis, $I^n \cdot I^n = V \ni 0$, $T^n \cdot A_0 = T = V$

$U = I$ T = fin. gen. set of I

$T \cdot I = I^2 \subset I$ $(T) = I$

2.) $\Rightarrow$ 1.)

U, and T as in def.

$A_0 = \mathbb{Z} + U$

A_0 = subring $U^n \subset U$

$A_0 \cdot A_0 \subset (\mathbb{Z} + U)(\mathbb{Z} + U)$

$\subseteq \mathbb{Z} + U + U^2 \subseteq \mathbb{Z} + U$ ✓

open: $\mathbb{Z} + U = \bigcup_{n \in \mathbb{Z}} (n + U)$

bounded: $V \ni 0$

$\exists n$: $U^n \subset V$

$U^n \cdot A_0 = U^n (\mathbb{Z} + U) \subseteq U^n + U^{n+1} \subset U^n$ ✓

A = top. ring

$V \subset A$ open subgroup

$T \subset U$ fin. subset

$T \cdot V = U^2 \subset U$, $(U^n)_{n=1}^{\infty}$ = basis of open n.

$A_0 \subset A$ subring, open, bounded.

$\Rightarrow$ A_0 is a ring of def in A = f adic.

$T(n) \cdot U^n \subset A_0$ = open

basis

$T(n) = \{ t_1 \cdots t_n \mid t_i \in T \}$

$I = (T(n)) \trianglelefteq A_0$

fin. gen. ideal.

(A_0, I) = pair of def.

I defines the top on A_0

I^n = open, form a basis

$I^n \supseteq U^{n+r}$ $T(n) \cdot U$

$(T(n)) = (T(n) \cdot A_0)^n$

$I^n = T(nr) \cdot A_0 \supset T(nr) \cdot U^{nr}$

$\supseteq U^{nr+r}$

$U^m A_0 = U^m$ $I^{N_m} \subset U^m$

Cor: A = f adic ring

(i) $A_0 A_1$ = rings of def $\Rightarrow$ $A_0 \cap A_1$, $A_0 \cdot A_1$ are also rings of def.

(ii) $B \subset C$ are subrings of A, B = bounded, C = open

$\Rightarrow$ $\exists$ ring of def. B' s.t. $B \subset B' \subset C$ $A = C$

(iii) $C \subset A$ is open $\Rightarrow$ C = f adic (w.r.t. subspace top)

A_0 ring of defn.

(A_0, I) = pair of def in A

$(C \cap A_0, I^n)$ $\Leftarrow$ adic ring + open

$I^n \subset C$ = open

$(I^m)^n \subseteq I \subseteq U$

by (ii), C = adic

$\Rightarrow$ $A = C$ w.l.o.g.

what we really have to show is that $\forall$ bounded ring is contained by a ring of defn.

A_0 = ring of def =

$A_0 \cdot B$ = ring, open, bounded

$\Rightarrow$ ring of defn.

power bounded and top nilp elements

def: $a \in A$ = top ring

a is power bounded if $\bigcup_{n=1}^{\infty} \{a^n\}$ = bounded

a = top nilp if $\forall$ open $V \ni 0$ $\exists n$: $a^m \in V$ if $m \geq n$.

E = subset $\subset A$

E = power-bounded if $\bigcup_{r=1}^{\infty} E(r)$ = bounded

E = top nilp if $\forall$ open $W \ni 0$ $\exists N \in \mathbb{N}$: $E(n) \subset W$ $\forall n \geq N$

$E = \{a\}$ these defn. are the same.

Example: $A = \mathbb{C}$ ($\mathbb{C}_p$)

power bounded: $|z| \leq 1$

top nilp: $|z| < 1$

Lemma: A = non-arch. top ring

$T \subset A$ a subset.

Equivalent:

1.) T' = subgroup gen by T

T = is bounded | power b | top nilp $\iff$ T' is bounded | power bounded | top.

2.) T = power bounded $\iff$ the ring gen. by T is bounded.

$T' \supseteq T$

T = bounded | p.b. | ...

$\Rightarrow T'$ ...

subgroup = $U \in \mathcal{O}$ open n.

subgroup $V \subset \mathcal{O}$ open n.

$V \cdot t \subset U$ $v \in V$ $t \in T$

$V \cdot T \subset U$ is closed under U

$V \cdot T' \subset U$ ✓

the ring gen. by T is subgroup gen.

$\bigcup_{r=1}^{\infty} T(r)$ ✓

Lemma: A = adic ring.

$x \in A$. Equiv:

1.) x is top. nilp.

2.) $\exists$ ideal of defn: $x^n \in I$

$I \trianglelefteq A$ s.t. x in A/I is nilp.

3.) $\exists$ $I \trianglelefteq A$ ideal of defn. s.t. $x \in I$

$\mathbb{Q}_p \supset \mathbb{Z}_p$

f adic adic

$x^n \in I \trianglelefteq A$

$J = I + xA \trianglelefteq A$

open open

$J^n \subset I \subseteq J$

$J^n = (I + xA)^n \subseteq I + x^n \cdot A \subseteq I + I \subset I$

3.) $\Rightarrow$ 1.)

$x \in I$ = ideal of defn.

$x^n \in I^n$ $\Rightarrow$ $x^m \in I^n$ $m \geq n$

$U \ni 0$ $\Rightarrow$ x = top. nilp.

A^{oo} = top nilp. elements in A

Cor: A = adic ring

$A^{oo} \trianglelefteq A$, radical, $\sqrt{J} = J$

A^{oo} = Union of all ideals of defn.

A^{oo} = ideal of defn $\iff$ A^{oo} is nilp. in A/I as an ideal for some I ideal of defn.

$J \trianglelefteq A$. $\sqrt{J}$ =

$\{ x \in A \mid \exists n \in \mathbb{N}_{>0}: x^n \in J \}$

= ideal of J

$\sqrt{A^{oo}} = A^{oo}$

x^n = top. nilp. $\Rightarrow$ x is top. nilp

x^n A/I nilp.

$x^{n \cdot m} \equiv 0$ (I) $\Rightarrow$ $x^{nm} \equiv 0 (I)$

$\sqrt{A^{oo}} = A^{oo}$

$x, y \in A^{oo}$ $(x+y) \in A^{oo}$

x, y nilp. A/I

$x+y$ — " — A/I

$x + y$ top. nilp.

$A^{oo} \trianglelefteq A$.

$I \trianglelefteq A$ ideal of defn.

$(A^{oo})^n \subset I \iff$

A^{oo} is nilp. in A/I

$A^{oo} = (A^{oo})^{nm} \subseteq I^M \subset U$

A^{o}: set of power bounded

Prop: A = f adic ring

A^{o} = open subring of A, int. closed, = $\bigcup$ of all rings of defn. in A

$A^{oo} \trianglelefteq A$ radical ideal.

$x \in A$ is p. bounded $\iff$

$\langle x \rangle$ = ring gen. by x = bounded

Lemma: $\langle x \rangle \subseteq$ ~~ideal of~~ ring of defn.

$A^{o} \subseteq \bigcup$ rings of defn.

$\forall$ ring of defn is bounded

$\Rightarrow$ $\forall$ $x \in$ ring of defn, is power

$\Rightarrow x \in A^{o} \Rightarrow A^{o} \supseteq \bigcup$ rings of defn.

def. a map $f: A \to B$ (A, B = top. rings) is bounded if it maps bounded sets to bounded sets.

it is not true in gen. that cont. homo of f adic rings is bounded.

def: A, B = f adic rings

a ring homo $f: A \to B$ is adic if $\exists$ (A_0, I) pair of defn. in A and a ring of defn. $B_0 \subset B$ s.t.

$f(A_0) \subseteq B_0$, $f(I) \cdot B_0$ = ideal of defn. in B_0.

Claim: 1.) $\forall$ adic morph is cont + bounded

2.) if $f: A \to B$ is surj + cont. $\Rightarrow$ f = adic

Top: $\mathbb{Q}_p$ $\mathbb{Q}[X]/(x^2)$