Rigid analytification and uniformization

Jef Laga

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1 Analytification

1.1 Over the complex numbers

Definition 1.1. *A complex analytic space is a locally ringed space $(X, \mathcal{O}_X)$ such that for each $x \in X$ there exists an open $U \subset X$ with the property that $(U, \mathcal{O}_X|_U)$ is isomorphic to $\{v \in V \mid f_1(v) = \cdots = f_k(v) = 0\}$, where V is an open subset of $\mathbb{C}^n$ and f_i are complex analytic functions (in other words, holomorphic functions) defined on V.*

Let X be a finite type scheme over $\mathbb{C}$. Then X is locally given by $\operatorname{Spec} \mathbb{C}[t_1, \ldots, t_n]/(f_1, \ldots, f_k)$. It follows that $X^{an} := X(\mathbb{C})$, endowed with the analytic topology and analytic structure sheaf, is a complex analytic space, called the analytification of X. Moreover, in the category of locally ringed spaces (in which both schemes and complex analytic spaces fully faithfully embed), we have a morphism $\epsilon \colon X^{an} \to X$ induced by the inclusion $X^{an} \subset X$. Given any other complex analytic space Z and morphism $Z \to X$, it uniquely factors through $X^{an} \to X$. We therefore find the following characterization of the analytification:

Lemma 1.2. *Let X be a finite type scheme over $\mathbb{C}$. Then the functor of points of X^{an} is given by*

$$\begin{aligned}(\textit{Complex analytic spaces}) &\to (\textit{Sets})\\ Z &\mapsto \operatorname{Hom}(Z, X).\end{aligned}$$

The Hom*-set is taken in the category of locally ringed spaces.*

This defines a functor:

$$(-)^{an} \colon (\text{Finite type schemes}/\mathbb{C}) \to (\text{Complex analytic spaces}).$$

It has the following fundamental properties:

1. X is connected/reduced/smooth/separated/proper if and only if the same is true for X^{an}. (In more common parlance, replace separated by Hausdorff and proper by compact.)
2. If X is proper, the functor $\mathscr{F} \mapsto \mathscr{F}^{an} = \epsilon^* \mathscr{F}$ from coherent sheaves on X to coherent sheaves on X^{an} is an equivalence and induces an isomorphism on sheaf cohomology groups.
3. The functor $(-)^{an}$ is fully faithful when restricted to proper schemes over $\mathbb{C}$.
4. If Z is a compact Riemann surface, there exists a smooth projective connected curve $X/\mathbb{C}$ with $X^{an} \simeq Z$.

1.2 In the rigid setting

Let k be a non-archimedean field. Then the story is very similar.

Definition 1.3. *Let X be a finite type scheme over k. An* analytification *X^{an} of X is a rigid analytic space over k with functor of points*

$$\begin{aligned}(\textit{Rigid analytic spaces}) &\to (\textit{Sets}) \\ Z &\mapsto \operatorname{Hom}(Z, X).\end{aligned}$$

The Hom*-set is taken in the category of locally G-ringed spaces.*

By definition, an analytification of X is unique up to unique isomorphism if it exists and comes with a natural map $\epsilon\colon X^{an} \to X$.

We now prove that X^{an} exists for any X/k of finite type.

Lemma 1.4. *Let X be an affine scheme of finite type over k and Z a rigid space. Then the natural map $\operatorname{Hom}(Z, X) \to \operatorname{Hom}(\mathcal{O}(X), \mathcal{O}(Z))$ is an isomorphism.*

Proof sketch. We construct an inverse to the above natural map. Since $\mathcal{O}_Z$ is a sheaf, we may assume that Z is affinoid. So let $X = \operatorname{Spec} A$ and $Z = \operatorname{Sp} B$, where A is a finitely generated k-algebra and B is a Tate k-algebra. Then the inverse is given by sending $\phi\colon A \to B$ to $\operatorname{Sp} B \to \operatorname{Spec} A, \mathfrak{m} \mapsto \phi^{-1}(\mathfrak{m})$. □

We construct the analytification of $\mathbb{A}^1_k$ explicitly. Let $c \in k$ be an element with $|c| > 1$. Let $T(c^n) = k\langle c^{-n}x\rangle$, seen as a sub-algebra of $T(1) = k\langle x\rangle$. The chain

$$\cdots \subset T(c^2) \subset T(c) \subset T(1)$$

gives rise to a chain of inclusions of closed balls

$$D(1) \subset D(c^1) \subset D(c^2) \subset \cdots$$

By glueing rigid analytic spaces, we obtain the rigid space $\mathbb{A}^{1,rig}_k$, the rigid analytic affine line.

Lemma 1.5. *The rigid space $\mathbb{A}^{1,rig}_k$ is the analytification of $\mathbb{A}^1_k$*

Proof. By Lemma 1.4, we have to show that $\operatorname{Hom}(Z, \mathbb{A}^{1,rig}_k) = \mathcal{O}_Z(Z)$ for any rigid space Z. It suffices to check this for $Z = \operatorname{Sp} A$ affinoid. Since $\operatorname{Sp} A$ is affinoid and the cover $\cup_{n\geq 0} D(c^n)$ of $\mathbb{A}^{1,rig}_k$ is admissible, any morphism $\operatorname{Sp} A \to \mathbb{A}^{1,rig}_k$ lands in $D(c^n)$ for some $n \geq 0$. To such a morphism corresponds an element $f \in A$ of norm $|f| \leq |c|^n$. Conversely, given an element $f \in A$ of norm $|f| \leq |c|^n$, there exists an associated morphism $\operatorname{Sp} A \to \mathbb{A}^{1,rig}_k$ mapping into $D(c^n)$. Since any element of A is of norm $\leq |c|^n$ for some $n \geq 0$, this completes the proof. □

Lemma 1.6. *Suppose that $X = \operatorname{Spec} k[t_1, \ldots, t_n]/(f_1, \ldots, f_k)$. Then X^{an} exists.*

Proof. Let $c \in k$ be an element with $|c| > 1$. Let $A_m = k\langle c^{-m}x_1, \ldots, c^{-m}x_n\rangle/(f_1, \ldots, f_k)$ and $U_m = \operatorname{Sp} A_m$. By the same reasoning as Lemma 1.6, $X^{an} = \cup_{m\geq 0} U_m$ is the analytification of X. □

Proposition 1.7. *Let X/k be of finite type. Then X^{an} exists.*

Proof. If X is affine, this follows from the previous lemma. To glue the affine patches, one needs the following fact: if X has an analytification $\epsilon\colon X^{an} \to X$ and $U \subset X$ is an open subscheme, $U^{an} = \epsilon^{-1}(U)$ is an analytification of U. □

This defines a functor:

$$(-)^{an}\colon (\text{Finite type schemes}/k) \to (\text{Rigid analytic spaces}/k).$$

Theorem 1.8. *The functor $(-)^{an}$ satisfies the same properties 1-4 as in the complex case.*

Note that we have not defined what it means for a rigid space to be connected/reduced/smooth/separated or proper. These definitions are straightforward, except for defining properness which requires some work. The proof is probably very similar!

2 The Tate curve

2.1 The story of $\mathbb{C}$

Let τ be an element of the upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ and let $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$. Then $\mathbb{C}/\Lambda_\tau$ is the analytification of the elliptic curve $E_\tau/\mathbb{C}$ with Weierstrass equation

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau).$$

Here

$$g_2(\tau) = 60 \sum_{\lambda \in \Lambda_\tau \setminus \{0\}} \frac{1}{\lambda^4},$$
$$g_3(\tau) = 140 \sum_{\lambda \in \Lambda_\tau \setminus \{0\}} \frac{1}{\lambda^6}.$$

Explicitly, the map $\mathbb{C} \to E_\tau(\mathbb{C})$ is given by $z \mapsto (\wp(z,\tau), \wp'(z,\tau))$, where $\wp(z,\tau)$ is the Weierstrass $\wp$-function

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda_\tau \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

and where $\wp'(z,\tau)$ is the derivative of $\wp(z,\tau)$ with respect to z.

Since $\Lambda_\tau = \Lambda_{\tau+n}$ for all $n \in \mathbb{Z}$, we may introduce $q = e^{2\pi i \tau}$ and write g_2, g_3 as functions in q. After some substitutions, we see that E_τ is isomorphic to

$$E_q\colon y^2 + xy = x^3 - a_4 x - a_6 \tag{2.1.1}$$

Where

$$a_4 = 5 \sum_{n \geq 1} \frac{n^3 q^n}{1 - q^n} = 5q + 45q^2 + 140q^3 + \dots$$
$$a_6 = \sum_{n \geq 1} \frac{7n^5 + 5n^3}{12} \cdot \frac{q^n}{1 - q^n} = q + 23q^2 + 154q^3 + \dots$$

Miracle: the power series $a_4(q), a_6(q)$ lie in $\mathbb{Z}[[q]]$.

Let $\Delta(q)$ be the discriminant of the above equation. It turns out that $\Delta(q) = q\prod(1-q^n)^{24}$.

Definition 2.1. *We call the curve with Equation* (2.1.1) *the* Tate curve. *It is an elliptic curve over* $\mathbb{Z}[[q]][\Delta(q)^{-1}]$.

Using the exponential map we see that $\mathbb{C}/\Lambda_\tau \simeq \mathbb{C}^\times/q^{\mathbb{Z}}$. Moreover $|q| < 1$ since $\tau \in \mathcal{H}$. Writing $u = e^{2\pi i z}$ and writing $\wp(z,\tau)$ and $\wp'(z,\tau)$ in terms of u and q, we obtain:

Proposition 2.2. *Let $q \in \mathbb{C}^\times$ with $|q| < 1$. Then the complex analytic space $\mathbb{C}^\times/q^{\mathbb{Z}}$ is the analytification of the elliptic curve E_q given by Equation* (2.1.1)*. The isomorphism is given by $\mathbb{C}^\times \to E_q(\mathbb{C}), u \mapsto (X(u,q), Y(u,q))$, where*

$$X(u,q) = \sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2\sum_{n\geq 1} \frac{nq^n}{1-q^n}$$

$$Y(u,q) = \sum_{n\in\mathbb{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + \sum_{n\geq 1} \frac{nq^n}{1-q^n}$$

Moreover, every elliptic curve over $\mathbb{C}$ arises in this way.

2.2 The non-archimedean case

Let k be a p-adic field. The main point of the Tate curve is the following theorem, which does not need rigid spaces to state.

Theorem 2.3. *Let $q \in k^\times$ be an element satisfying $|q| < 1$. Then $a_4(q), a_6(q)$ (defined in the previous section) are elements of k, and E_q is an elliptic curve over k. There exists an isomorphism*

$$\bar{k}^\times/q^{\mathbb{Z}} \simeq E_q(\bar{k})$$

compatible with the Galois action on both sides. The curve E_q has split multiplicative reduction. Conversely, any elliptic curve over k with split multiplicative reduction arises in this way, i.e. is of the form E_q for some q.

Sketch of proof. Since $a_4, a_6 \in \mathbb{Z}[[q]]$ and $|q| < 1$, it is clear that $a_4(q), a_6(q)$ converge to elements of k. To write down an isomorphism $\bar{k}^\times/q^{\mathbb{Z}} \simeq E_q(\bar{k})$, one writes down formulas for the isomorphism $\mathbb{C}^\times/q^{\mathbb{Z}} \to E_q(\mathbb{C})$ using the Weierstrass $\wp$-function and see (amazingly!) that they still work. The curve E_q has split multiplicative reduction by reducing Equation (2.1.1). Moreover the j-invariant of E_q is the *actual* j-function $j(q) = q^{-1} + 744 + \ldots$. It is elementary that $q \mapsto j(q)$ induces a bijection

$$\{q \in k^\times \mid |q| < 1\} \to \{j \in k \mid |j| > 1\}.$$

Let E/k be an elliptic curve with split multiplicative reduction. Then $|j(E)| > 1$ so E is isomorphic to E_q over $\bar{k}$ for some $q \in k$ with $|q| < 1$. Moreover $|j(E)| > 1$ implies that $j(E) \neq 0$ or 1728. Therefore E and E_q are quadratic twists of each other. Since there is a unique twist where the reduction is split, we conclude that $E \simeq E_q$ over k, as claimed. □

Corollary 2.4. *Let E/k be an elliptic curve with split multiplicative reduction. Let $l \neq p$ and let $T_l E$ be the l-adic Tate module of E. Then there is a short exact sequence of Galois modules*

$$1 \to \mathbb{Z}_l(1) \to T_l E \to \mathbb{Z}_l \to 1.$$

In other words, $T_l E$ has a $\mathbb{Z}_l$-basis in which the Galois action is of the form $\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$, where χ is the cyclotomic character.

2.3 The Tate curve as a rigid space

Let $q \in k$ with $|q| < 1$. We will now introduce a rigid space $X_q = \mathbb{G}_{m,k}^{an}/q^{\mathbb{Z}}$ and upgrade the above theorem to an isomorphism $X_q \simeq E_q^{an}$. We first need to talk about quotients of rigid spaces.

Definition 2.5. *Let Γ be a group acting freely and continuously on a rigid space X. We say the action is properly discontinuous if there exists an admissible covering of the form $\{\gamma \cdot U_i\}_{\gamma \in \Gamma, i \in I}$ of X with $\gamma \cdot U_i \cap U_i = \varnothing$ unless $\gamma = 1$ and such that the sets $\cup_{\gamma \in \Gamma} \gamma \cdot U_i$ are admissible for all $i \in I$.*

In the above situation, we can form the quotient $Y = X/\Gamma$. Indeed, to construct Y, by glueing it suffices to give a cover by rigid spaces V_i, admissible opens $V_{ij} \subset V_i$ and comparison isomorphisms $V_{ij} \xrightarrow{\sim} V_{ji}$ satisfying some cocycle compatibility. We may take $V_i = U_i$ and $V_{ij} = U_i \cap (\cup \gamma \cdot U_j)$ and $V_{ij} \to V_{ji}$ the natural isomorphism. One can check that this rigid space comes with a Γ-invariant map $X \to Y$ (where Γ acts trivially on Y), and that any Γ-invariant map $X \to Z$ factors through $X \to Y$.

We first explicitly write down $\mathbb{G}_{m,k}^{an}$. Since $\mathbb{G}_{m,k} = \operatorname{Spec} k[u,v]/(uv-1)$, we know by construction that $\mathbb{G}_{m,k}^{an}$ has an admissible cover

$$\cup_{n \geq 0} \{|u|, |v| \leq |q|^{-n}\} = \cup_{n \geq 0} \{|q|^n \leq |u| \leq |q|^{-n}\}.$$

More generally, if $a \leq b$ are rational numbers let $X[a,b]$ be the subset of $\mathbb{G}_{m,k}^{an}$ given by $\{|q|^b \leq u \leq |q|^a\}$. Then $X[a,b]$ is open affinoid. Moreover the above cover is $\cup X[-n,n]$.

Let $t_q \colon \mathbb{G}_{m,k}^{an} \to \mathbb{G}_{m,k}^{an}$ be the multiplication by q map. It sends $X[a,b]$ to $X[a+1,b+1]$. This defines a $\mathbb{Z}$-action on $\mathbb{G}_{m,k}^{an}$ which is free and continuous.

Lemma 2.6. *This action is properly discontinuous.*

Proof. We may take $U_0 = X[0,1/2]$ and $U_1 = X[1/2,1]$. Then the cover $\{\gamma \cdot U_i\} = \{X[n, n+1/2]\} \cup \{X[n+1/2, n+1]\}$ is an admissible cover that satisfies the requirements for being properly discontinuous. $\square$

It follows that the quotient $X_q = \mathbb{G}_{m,k}^{an}/q^{\mathbb{Z}}$ exists as a rigid space; write $\pi \colon \mathbb{G}_{m,k}^{an} \to X_q$ for the quotient map. Explicitly, it is obtained by glueing $X[0,1/2]$ and $X[1/2,1]$ along their boundary: $X[0,0] \sqcup X[1/2,1/2] \xrightarrow{t_q \sqcup \mathrm{Id}} X[1,1] \sqcup X[1/2,1/2]$. Using this description or the universal property of quotients, we see that the structure sheaf of X_q is given by

$$\mathcal{O}_{X_q}(U) = \{f \in \mathcal{O}_{\mathbb{G}_{m,k}^{an}}(\pi^{-1}U) \mid t_q^* f = f\}$$

For example, $\mathcal{O}_{X_q}(X_q)$ consists of those element $\sum_{n \in \mathbb{Z}} a_n t^n$ with $|a_n|\rho^n \to 0$ as $|n| \to +\infty$ for all $\rho > 0$ satisfying $a_n = q^n a_n$. Therefore $a_n = 0$ if $n \neq 0$, so $\mathcal{O}_{X_q}(X_q) = k$.

It is clear that for every finite field extension l/k we have $X_q(l) = l^\times/q^{\mathbb{Z}}$.

Proposition 2.7. *There exists an isomorphism $X_q \simeq E_q^{an}$ of rigid spaces.*

Proof. The one used in the above theorem is analytic. $\square$