Topology of Berkovich Spaces

1) Topology of X^{an}

$X/k \rightsquigarrow X^{an} \leftarrow$ "analytic object"

non arch field

étale $\leftarrow$ X/K, K a # field $\rightsquigarrow$ $X(\mathbb{C})$ — sing, nice

$X/\mathbb{Q}$ smooth proper curve, then $\chi(X(\mathbb{C}))<0$
$\Rightarrow X(\mathbb{Q})$ is finite

Mirror this: $X^{an} \leftarrow$ "nice topological man"

1) X^{an} always exists — 1st
explicitly describe

2) X^{an} is "nice" hausdorff, loc. contractible
X^{an} reflects "X"
eg: X/k is proper, X^{an} is compact

3) X^{an} has nice homotopy, $\exists S$ st. $X^{an} \sim S$ ← finite CW complex

Analytification

1) A (good) Berkovich space is a pair $(X, \mathcal{O}_X)$
X top. space, $\mathcal{O}_X$ is a sheaf st. this is a LRS
and $(X, \mathcal{O}_X)$ is locally isomorphic to $(\mathcal{M}(A), \mathcal{O}_A)$

affinoid algebra

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Thm X/k is a variety, then X^{an} is good

Def^n If X/k is a variety, then X^{an} is a Berkovich space s.t.

$\exists\ X^{an} \longrightarrow X$, and if Z is a BS s.t. $Z \to X$

$$\begin{tikzcd} & X^{an} \arrow[dr] & \\ Z \arrow[ur, dashed] \arrow[rr] & & X \end{tikzcd}$$

this commutes

X^{an} represents the Berk functor

$$Z \longrightarrow \mathrm{Hom}_{LBS}(Z, X)$$

$$A \quad , \quad \mathcal{M}(A) \text{ topology}$$

1) $V \subseteq \mathcal{M}(A)$ s.t. $V \cong \mathcal{M}(V')$

$$\mathcal{O}_A(V) = V'$$

2) If $V = \bigcup_{i=1}^{n} \mathcal{M}(A_i)$ for A_i affinoid /

V special + $\mathcal{O}_A(V) = \ker\left(\prod_i A_i \longrightarrow \prod_{i,j} A_{i \cap j}\right)$

$$\mathcal{O}_X(U) = \varprojlim_{\text{special } V \subseteq U} A_i \quad \leftarrow \text{ Tate acyclicity}$$

Does it exist? What does it look like?

Wouter
/ Zomervrucht

Case 1 $X = \mathbb{A}^n_k$, $X^{an} = \mathbb{A}^{n,an}_k = \{$ mult. seminorms $k[T_1, \dots, T_n]\}$

on $(\mathbb{A}^{1,an}_k)^n$

$X^{an} \longrightarrow X$? $|\cdot|_x : A \longrightarrow \mathbb{R}_{\geq 0}$

$|\cdot|_x \longmapsto \ker(|\cdot|_x)$ $\{f \in k[T_1, \dots, T_n] : |f|_x = 0\}$

$\in \operatorname{Spec}(k[T_1, \dots, T_n])$

ex this is an analytic function

Case 2

Suppose S/k is a variety s.t. S^{an} exists

$T \hookrightarrow S$ as an open immersion

Claim T^{an} exists. $- \times -$ in LRS

Proof $T^{an} := \underbrace{T \times_S S^{an}}_{\text{open subset of } S^{an} \text{ so is clearly good Berkovich}}$

$$\begin{tikzcd}
Z \arrow[r, dashed, "\exists!"] \arrow[rr, dashed, bend left] \arrow[dr, bend right] & T \times_S S^{an} \arrow[r] \arrow[d] & S^{an} \arrow[d, "\pi"] \\
& T \arrow[r, hook] & S
\end{tikzcd}$$

Case 3 $T \hookrightarrow S$ closed immersion

T is defined by $\mathcal{I}$ sheaf of ideals of $\mathcal{O}_S$

doesn't $\pi^* \mathcal{I}$ sheaf of ideals of $\mathcal{O}_{S^{an}}$

$\therefore$ Defines a closed immersion

$$\text{"}T^{an}\text{"} \hookrightarrow S^{an}$$

$$\begin{tikzcd} \text{"}T^{an}\text{"} \arrow[r, hook] \arrow[d, dashed] & S^{an} \arrow[d, "h"] \\ \tilde{T} \arrow[r, hook] & S \end{tikzcd}$$, T^{an} is a Berk. space

Case 4 If X is affine, then $\exists$ closed immersion $X \hookrightarrow \mathbb{A}^n_k$ for some n

By case 3, 1, X^{an} exists

Case 5 For general X, take case 4, case 2 allows gluing

Rem Entirely based on $\mathbb{A}^n_k$ + categories

$$\mathbb{A}^{n\,an}_k \xrightarrow{\pi} \mathbb{A}^n_k$$

$$\text{(1)}$$

$$\{(x,w) : w : K(x) \to \mathbb{R}_{\geq 0}\} \longmapsto \{x \in \mathbb{A}^n_k\}$$

$$|\cdot|_x \longleftrightarrow (\ker(|\cdot|_x), |\cdot|_x)$$

Rem $X^{an} = \{(x,w) : x \in X,\ w : K(x) \to \mathbb{R}_{\geq 0}\}$

Rem w extends the norm on k,

weakest top s.t. π is cts and $\forall f \in k[T_1, \dots, T_n]$

$$(x,w) \longmapsto w(f(x)) \text{ is cts.}$$

On X^{an}, what topology do we give

if X is affine, do this, $X = \mathrm{Spec}(A)$

$X^{an} \to X$ is cts and $\forall f \in A$

$$(x, \omega) \mapsto \omega(f(x)) \text{ is cts}$$

X^{an} is not affine

$X^{an} \to X$ is cts, $\forall U \subseteq X$ open, $\forall f \in \mathcal{O}_X(U)$

$$\boxed{\begin{aligned} \pi^{-1}(U) &\to \mathbb{R}_{\geq 0} \\ (x, \omega) &\mapsto \omega(f(x)) \text{ is cts.} \end{aligned}}$$

Def^n X^{an} is the topological space

$\{(x, \omega)\}$: $\begin{aligned} X^{an} &\to X \\ (x,\omega) &\mapsto x \end{aligned}$ is cts

and $\forall U \subseteq X$ open, $\forall f \in \mathcal{O}_X(U)$

$$\begin{aligned} \pi^{-1}(U) &\to \mathbb{R}_{\geq 0} \\ (x, \omega) &\mapsto \omega(f(x)) \quad \text{is cts.} \end{aligned}$$

Rem $X^{an} \to X$ is surjective as a map of sets.
It's murky + ZL

2 **What properties does it have?**

Thm A is a ring, $\mathrm{Spec}(A)^{an}$ is Hdorff.

Proof $|\cdot|_x \neq |\cdot|_y$ mult. snorms on A $\quad |\cdot|_x : A \to \mathbb{R}_{\geq 0}$

$\exists f : |f|_x < r < |f|_y$

$U_x := \{ |\cdot|_z : |f|_z < r \}$ $\quad U_y := \{ |\cdot|_z : |f|_z > r \}$

$\ni |\cdot|_x$ $\quad \ni |\cdot|_y$

Cor X^{an} is locally hdorff for any X

Lemma T is hdorff iff loc. hdorff + separated

Proof Ex.

"$(X^{an} \to \mathrm{Spec}(k)^{an})$"

$\mathrm{Spec}(k)$

Lemma X/k is variety, then X^{an} is

Proof X^{an} is separated (as a LRS) iff $X^{an} \to X^{an} \times_{\mathrm{Spec}(k)} X^{an}$ is closed

$x \mapsto (x, x)$

Quasi compact $X^{an} \times X^{an} = (X \times_k X)^{an}$

X/k is separated. $X \to X \times_k X$ is a (finite) closed immersion

$X^{an} \to (X \times_k X)^{an}$ is a closed immersion

Cor X^{an} is hausdorff.

Lemma $\varphi: X \to Y$ a morphism of k varieties, s.t. is set theoretically surjective

$\varphi^{an}: X^{an} \to Y^{an}$ is also surjective

Proof $\pi: X^{an} \to X$, $y \in Y^{an}$, $K(\pi_1(y)) \hookrightarrow K(y)$

$$(Y^{an})_y \cong (\underbrace{\pi(y) \times_{K(\pi(y))} K(y)}_{\text{sch}})^{an}$$

Corollary φ is surjective iff all the fibres are non empty

Fibres of φ give us fibres of φ^{an} so they're nonempty.

Thm X/k is proper, then X^{an} is compact

Proof **Ex** $(\mathbb{P}_k^n)^{an}$ is compact

Cor 2 X/k is projective: $X \hookrightarrow \mathbb{P}_k^n$

$(X)^{an} \hookrightarrow \mathbb{P}_k^{n,an} \leftarrow$ compact

is compact

Cor 3 X/k is proper. Chow's lemma $\exists X'$ s.t. X'/k is projective

and $X' \to X$ is surjective (and projective)

$(X')^{an}$ is compact

and maps surjectively to $X^{an} \leftarrow$ is compact

NB $f: X \to Y$ is flat, unramified / smooth / separated / injective / open immersion / surjective / finite type

then f^{an}

Moreover, if f is of finite type, f is dominant / closed immersion / proper iff f^{an} is.

"GAGA" 90, 93, section 3

Lemma Every connected Berkovich space is arcwise connected

$x \neq y$, $\exists\ \varphi: [0,1] \to X$
s.t. $\varphi(0) = x$, $\varphi(1) = y$, φ is homeomorphism onto image

Proof (sketch, path connectedness)

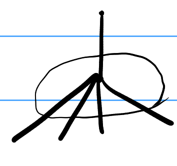
X is good, $x \in X$ so assume is k-affinoid.

By extending k to K, X is strictly k-affinoid
$X_K \to X$ is surjective
path conn. $\quad$ p. conn.

Noether normalization $X = E_k^n$ is unit polydisc

$X \to E_k^n$
path connected. $\quad$ path connected

E_k^1 is path connected



Assume E_k^{n-1} is pc

$$\pi : E_k^n \longrightarrow E_k^{n-1}$$
$$\qquad\qquad\qquad\qquad \psi_x$$

$$\pi_x \cong E^1_{K(x)}$$

$$\sigma : E_k^{n-1} \to E_k^n$$
$$x \longmapsto (T) \leftarrow \mathcal{M}(\overline{K(x)\{T\}})$$

$y_0, y_1 \in E_k^n$

$$x_0, x_1 := \sigma(\pi(y_i))$$

$$y_0 \to x_0 \to x_1 \to y_1$$

Thm Berkovich spaces are locally contractible

Berkovich only proves for smooth, 93 page paper.

1 page proof: Hrushovski - Loeser $\leftarrow$ totally different

If X/k is finite type + k has a countable + dense subset

eg: $k = \mathbb{Q}_p$, $k = \mathbb{C}((t))$
$\quad\ \ \mathbb{Q} \qquad\qquad \mathbb{Q}(t)$

Then $X^{an} \hookrightarrow \mathbb{R}^n$ for some n (+ Poonen)

First thing $H^i(|X^{an}|, \mathbb{Q}_\ell) = H^i_{ét}(X, \mathbb{Q}_\ell)_0^{sm}$ $\leftarrow$ Berkovich

$\ell \neq p$, $\ell = p$

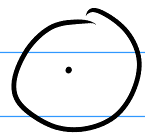
Thm X/k variety, X^{an}

$\exists\, S \subseteq X^{an}$, s.t. S is a finite CW complex

and $\exists\, X^{an} \longrightarrow S$

$X^{an} \times [0,1] \to X^{an}$

$\mathbb{R}^2 \setminus \{0\}$



$X^{an} \times \{0\} = id$

$S \times \{t\} = id$

$X^{an} \times \{1\}$ is a homotopy equivalence $X^{an} \to S$

— Moreover, if G is a finite group $\circlearrowright X$

Pick S to be a G-CW complex —

and $X^{an} \times [0,1] \to X^{an}$ is G-equivariant

$(X^{an})^G$ $\qquad$ S^G is non-empty

algebraic topology

- Using $\mathcal{X}/\mathcal{O}$ a model for X/k

$\mathcal{O}$ is the ring of integers

$\mathcal{X}_k \cong X$, $\mathcal{X}_{\mathbb{F}}$ is a strict normal crossing divisor

$|\Delta(\mathcal{X}_{\mathbb{F}})| \leftarrow$ is our choice for S

$\downarrow$

X^{an} is a homotopy equivalence

$\mathcal{X}/\mathcal{O}$ is a snd model

$\mathcal{X}_\pi = \bigcup_{i \in I} E_i$, E_i is an irred. comp.

d simplices are given $\pi_0(E_J)$

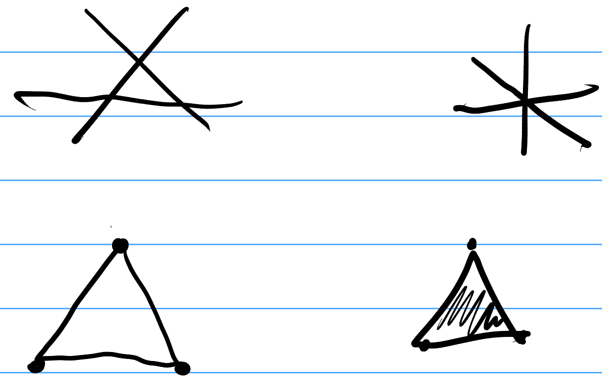
$E_J = \bigcap_{j \in J} E_j$, J runs through subsets of I . $|J| = d+1$

C a conn. comp. of E_J

C' $E_{J'}$

$\Delta_C \subseteq \Delta_{C'}$ iff $C' \supseteq C$

eg. $\mathcal{X}_\pi$



$$\begin{array}{ccccc} \Delta(\mathcal{X}_\pi) & \xrightarrow{=} & Sk(\mathcal{X}) & \subseteq & X^{an} \\ & & & & \downarrow \pi \\ & & & & X \end{array}$$

$\pi(Sk(\mathcal{X})) = \eta$ gen. pt.