The topology of Berkovich Spaces

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- The main thing that makes Berkovich spaces a good candidate for our non-archimedean space is that they have very nice topological properties.
- Doubley nice is that not only are they good topological spaces, but their topology often reflects what the topology of the underlying variety should be (ie: Hausdorff, proper varieties give us compact Berkovich spaces, etc).
- Tripley nice is that they have good homotopy types, and moreover, in certain cases, we have a model for their homotopy types.

I should mention that model theory is a seriously powerful tool when working with Berkovich Spaces, and a lot of results are down to Hrushikov and Loeser, who did a very model theoretic paper where they proved some big statements. For those of you who've run into model theory results before, the style of argument is just so different to algebraic geometry and number theory, and relies on a hugely different background, so I won't go into those proofs, and I'll only mention the results, then in some cases prove a weaker statement without the model theory.

1 Analytification

Definition 1.1. A *good Berkovich space* is a locally ringed space, $(X, \mathcal{O}_X)$ such that locally $(X, \mathcal{O}_X)$ looks like $(\mathcal{M}(\mathcal{A}_V), \mathcal{O}_{\mathcal{A}_V})$ for $\mathcal{A}_V$ an affinoid algebra.

This is Berkovich's original definition of a Berkovich space. It was later refined to include spaces that you can piece together via charts, which is a more analytic outlook, but since all analytifications of varieties are "good Berkovich spaces", I'll restrict to these for today.

Jef mentioned that the whole point of analytification is the following. If X is a scheme, then X^{an} is a Berkovich space with a map $X^{an} \to X$, such that if Z is any Berkovich space, with $Z \to X$ a map of locally ringed spaces, then there is a unique map $Z \to X^{an}$ such that the obvious diagram commutes.

That is: X^{an} is the the object of the category of Berkovich spaces that represents the functor $\mathrm{Hom}_{LRS}(-, X)$. This is kind of the point of them, and is arguably the best way to think about them, but doesn't really give us anything workable. It's also not clear that it exists, so I'll sketch how you construct it, which also gives us a description of the topological space.

From here, I'll fix k to be a non-archimedean valued field, $\mathbb{F}$ its residue field, and v its norm. For a k algebra, A, note that $\{$multiplicative seminorms on A extending the valuation on $k\}$ is the same as the set $\{(x, w) : x \in \mathrm{Spec}(A), v \text{ a norm on } \kappa(x) \text{ extending } v\}$. This is because if $w : A \to \mathbb{R}$ is a multiplicative seminorm, then $\ker(w)$ is a prime ideal, and $w : A/\ker(w) \to \mathbb{R}$ is an actual norm.

We now sketch the construction of analytifications for general k varieties. (Proof taken from Wouter Zomervrucht's notes)

1. Case 1. $X = \mathbb{A}^n_k$. We have a candidate for the affinoid atlases: $X^{an} = \mathbb{A}^{n,an}_k = \{$multiplicative seminorms on $k[T_1, \ldots, T$ Moreover, the map $X^{an} \to X$ is given by $w \mapsto \ker(w)$. We also see that this is given the weakest topology such that $\mathbb{A}^{n,an}_k \to \mathbb{A}^n_k$ is continuous, and for every $f \in k[T_1, \ldots, T_n]$, the map $|\cdot|_x \mapsto |f|_x$ is also continuous.

2. Case 2: suppose that S is a k variety such that $\pi : S^{an} \to S$ exists, and let $T \to S$ be an open embedding. Then $T^{an} = \pi^{-1}(T)$ is a Berkovich space, and we can see easily that $T^{an} \to T$ is a Berkovich analytification, by universal properties.

3. Case 3: $T \subseteq S$ closed and $\pi : S^{an} \to S$ exists. Let $\mathcal{I}$ be the sheaf of $\mathcal{O}_S$ ideals defining T. Then $\pi^*\mathcal{I}$ is a sheaf of $\mathcal{O}_{S^{an}}$ ideals, which defines a closed subspace, T^{an}. We can see $T^{an} \to T$, and it is a berkovich analytification.

4. Case 4. If S is affine, then S is a closed subscheme of $\mathbb{A}^n$ for some n, so result holds by 1 and 3. The result then holds by gluing as in case 2.

Remark 1.2. For the purposes of this talk, I'm restricting to the variety case, so that $X \to \operatorname{Spec}(k)$ is finite type and separated. Analytification holds in more general contexts, but the description of the topological space doesn't work as nicely.

Remark 1.3. By our description of case 3, we can see that if $X = \operatorname{Spec}(A)$, then $X^{an} = \{$multiplicative seminorms on A exten
From earlier, we know that we can describe this as

$$\{(x, w) : x \in X, w : \kappa(x) \to \mathbb{R}_{\geq 0} \text{ extending v}\}.$$

Moreover, by our calculations on affine space, we see the topology on X^{an} is the weakest topology such that $X^{an} \to X$ is continuous, and for all $f \in A$, the map $(x, w) \mapsto w(f(x))$ is also continuous.

By the gluing case above, we can see that this definition generalises. That is, let X be a variety over k. Then X^{an} as a topological space is the set

$$\{(x, w) : x \in X, w : \kappa(x) \to \mathbb{R}_{\geq 0} \text{ extending v}\}.$$

Then the map $X^{an} \to X$ is given by $(x, w) \mapsto x$. The topology on X^{an} is the weakest such that $X^{an} \to X$ is continuous, and for all $U \subseteq X$ open and for all $f \in \mathcal{O}_X(U)$, the map

$$\begin{aligned} U^{an} &\to \mathbb{R}_{\geq 0} \\ (x, w) &\mapsto w(f(x)), \end{aligned}$$

is also continuous. From here, we'll adopt this as an alternative definition of the analytification.

Remark 1.4. It is possible to show $X^{an} \to X$ is surjective as a map of sets, though this takes some work and applications of Zorn's lemma.

2 Properties of Berkovich analytifications

We'd like to show that topological properties of the Berkovich analytifications.

Theorem 2.1. *Let A be a finitely generated k algebra. Then $\operatorname{Spec}(A)^{an}$ is an Hausdorff.*

Proof. Let $x, y \in \operatorname{Spec}(A)^{an}$, and think of x, y as multiplicative seminorms. Then there exists $f \in A$ such that $|f|_x \neq |f|_y$. Let r be such that $|f_x| < r < |f|_y$, and let $U = \{x \in \operatorname{Spec}(A)^{an} : |f|_x < r\}$ and $U' = \{x \in \operatorname{Spec}(A)^{an} : |f|_x > r\}$. This gives the result. □

Therefore, X^{an} is locally Hausdorff.

Lemma 2.2. *Let X/k be a variety. Then X^{an} is separated.*

Proof. Note, X^{an} is separated if and only if the diagonal subset is closed in in $X^{an} \times X^{an}$. It's a quick computation that $X^{an} \times X^{an} = (X \times_k X)^{an}$. Note that $X \to X \times_k X$ is a closed immersion, since X is separated. By the definitions of our analytifications earlier, analytification preserves closed immersions, so this holds. □

Corollary 2.3. *X^{an} is Hausdorff.*

Proof. Locally Hdorff + separated. ☐

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Lemma 2.4. *Let $\varphi : X \to Y$ be a morphism of k varieties. Then $X \to Y$ is surjective implies that $\varphi^{an} : X^{an} \to Y^{an}$ is surjective.*

Proof. Let $\pi : X^{an} \to X$. Let $x \in X^{an}$. Then we see that $k(\pi(x)) = \mathcal{O}_{X,\pi x}/\mathfrak{m}_{\pi(x)}$ is naturaly embedded in the residue field at x, $M(x)$, and there is an isomorphism:

$$(Y^{an})_x \cong (\pi(x) \otimes_{k(\pi(x))} M(x))^{an}.$$

We know that φ is surjective if and only if the fibres are non empty. Therefore φ^{an} is surjective if and only if φ is. ☐

Lemma 2.5. *Let X/k be proper. Then X^{an} is compact.*

Proof. It is a standard computation to see that $(\mathbb{P}^n_k)^{an}$ is compact (we have a pretty explicit description)

If X is projective, then we can show that X^{an} is compact, since there is a closed immersion $X^{an} \to (\mathbb{P}^n_k)^{an}$ and $(\mathbb{P}^n_k)^{an}$ is compact. For arbitrary X, Chow's lemma tells us that there exists a projective variety X'/k, and a projective, surjective, morphism $\psi : X' \to X$. We then have that $(X')^{an}$ is compact by the earlier work, and $(X')^{an} \to X^{an}$ is surjective, therefore X^{an} is compact. ☐

NB: The results sketched above hold in much more generality. In fact you can show that $f : X \to Y$ is flat/unramified/smooth/separated/injective/surjective/open immersion/finite type if and only if f^{an} is. Moreover, if f is of finite type, then f is dominant/closed immersion/proper if and only if f^{an} is.

Lemma 2.6. *Every connected Berkovich space is arcwise connected. That is: for x, y distinct points, there is a homeomorphism $\varphi : [0,1] \to X$ with $\varphi(0) = x$ and $\varphi(1) = y$.*

Sketch. For simplicity, assume that every point $x \in X$ has an affinoid neighbourhood (eg: analytification), so we can assume that X is k-affinoid, and by extending the ground field, we can assume that X_K is striclty K affinoid. Since $X_K \to X$ is surjective, we only need to check that it is path connected.

We can then use Noether normalisation to assume that $X = E^n_k$ is the unit polydisc (since there is a finite map $X \to E^n_k$, so E^n_k path connected implies X is.)

We can then show that E^n_k is pathwise connected. For $n = 1$, E_k is a tree, so that's alright. If $n > 1$, we project $\pi : E^n_k \to E^{n-1}_k$. Then the fibre over $x \in E^{n-1}_k$ is isomorphic to $E^1_{\kappa(x)}$. Note that we have a section of this projection: $\sigma : E^{n-1}_k \to E^n_k$, which maps $x \in E^{n-1}_k$ to $(T) \in E^1_{\kappa(x)} = \mathcal{M}(\kappa(x)\{T\})$.

Let $y_0, y_1 \in E^n_k$. Let $x_i = \sigma(\pi(y_i))$. The path from y_1 to y_2 is given by the collective paths $y_1 \to x_1 \to x_2 \to y_2$, which exist by induciton. ☐

The actual proof is hard and long.

The other main topological result, I'll only mention and won't even sketch.

Theorem 2.7. *Berkovich spaces are locally contractible.*

For smooth Berkovich spaces (eg: X^{an} where X is a smooth space), this was proven by Berkovich in his paper "smooth p-adic analytic spaces are locally contractible", and given the name of the paper, it's a bit much to cover in this talk). In general, this was proven by Hrushikov and Loeser, but the proof is incredibly model theoretic.

Also mention this one `http://www-math.mit.edu/~poonen/papers/berkovich.pdf`, but again probably very model theoretic.

3 Skeleta and homotopy types

I mentioned at the beginning that these spaces have nice homotopy types. The main theorem in this area is actually this one by Hrushikov and Loeser.

Theorem 3.1 (Theorem 14.2.1 of Hrushikov Loeser)**.** *Let X/k be a variety, and let X^{an} be its Berkovich skeleton. Then there exists a subset of X^{an}, S such that S is homeomorphic to a finite simplicial complex. Moreover, there is a strong deformation retraction from X^{an} onto S.*

Double moreover, if G is a finite group acting on X, this can be made G equivariant.

This theorem is very model theoretic in Hrushikov Loeser, and the proof isn't constructive and is based around model theory. Instead today, we'll give a construction of S in certain cases, and sketch a proof for this S.

Definition 3.2. Let $\mathcal{X}/\mathfrak{o}$ be a model for X/k. We say $\mathcal{X}$ is *sncd* if the special fibre is a strict normal crossings divisor of the model.

Definition 3.3. Let X/k be a variety and $\mathcal{X}/\mathfrak{o}$ be a model. Let $x \in X^{an}$, so that v is a valuation on $\kappa(y)$ where $x = (y, v)$. Let $\mathfrak{o}_x$ denote the valuation ring at x. Then define center(x), if it exists, to be the unique point that is the image of the closed point of the map $\mathrm{Spec}(\mathfrak{o}_x) \to \mathcal{X}$ that maps the generic point to y. Note center(x) always lies on the special fibre, by the definition of our valuations.

Definition 3.4. Let $\mathcal{X}/\mathfrak{o}$ be a model of X/k, and let E be an irreducible component of the special fibre with generic point ξ. Note that $\mathrm{center}_{\mathcal{X}}^{-1}(\xi)$ corresponds to a unique point, which we call the *divisorial point associated to* $(\mathcal{X}, E)$. This is because we see that $\mathcal{O}_{\mathcal{X},\xi}$ is a valuation ring on K, so gives us $w \in \mathrm{Val}_v(K)$, and since $\dim(\mathcal{O}_{\mathcal{X},\xi}) = 1$, it cannot strictly contain another valuation ring, so it must correspond to a valuation of rank 1.

We can explicitly describe the valuation as follows. Fix $\mathcal{X}$ and E, and let $w \in X^{an}$ be a divisorial point associated to $(\mathcal{X}, E)$. Let n denote the multiplicity of E in the divisor $\mathcal{X}_k$ on $\mathcal{X}$, and let $f \in \mathcal{O}_{\mathcal{X},\xi}$. Then we exactly have

$$w(f) = \frac{1}{n}\mathrm{ord}_E(f).$$

In general, we say $x \in X^{an}$ is divisorial if it is the divisorial point associated with some pair like above.

Let $\mathcal{X}/\mathfrak{o}$ be an sncd model of X, and let $E_1, \ldots, E_r$ be distinct irreducible components of the special fibre with multiplicities $N_1, \ldots, N_r$. Suppose $\bigcap_{i=1}^r E_i$ is non empty, and let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be such that $\sum_{i=1}^r \alpha_i N_i = 1$.

Let ξ be a generic point of $\bigcap_{i=1}^r E_i$, which, by definition of sncd, is regular and of pure dimension $n+1-r$.

Proposition 3.5. *There exists a unique real valuation*

$$w : K^\times \to \mathbb{R}$$

with valuation ring $\mathcal{O}_{\mathcal{X},\xi}$, such that $w(T_i) = \alpha_i$ for every i, and $T_i = 0$ for E_i in $\mathcal{X}$ at ξ, where T_i is a a function such that locally $E_i = \{T_i = 0\}$. Moreover, w is independent of the choice of T_is, and $w|_k = v$.

We'll prove this proposition slightly later. The proof is very long and commutative algebra.

Definition 3.6. We say that v is a *monomial point* if v is a monomial point associated with $(\mathcal{X}, E_1, \ldots, E_r, \alpha, \xi)$ for some choice of this data.

If you have monomial points x, x' corresponding to $(\mathcal{X}, A), (\mathcal{X}', A')$ (where A, A' are the rest of the data), we can always assume that $\mathcal{X} = \mathcal{X}'$, since the category of sncd models is cofiltered, so we can go to a model that dominates both $\mathcal{X}$ and $\mathcal{X}'$.

Remark 3.7. The idea behind that is that, by varying these αs, we can interpolate between divisorial points. This is important later when we introduce the skeletons.

The resk of this subsection is dedicated to the proof, and is largely commutative algebra.

Lemma 3.8. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let $(y_1, \dots, y_m)$ be a system of generators for $\mathfrak{m}_A$. Let $y_1, \dots, y_m$ be a system of generators for $\mathfrak{m}$, and let $\hat{A}$ denote the $\mathfrak{m}$-adic completion of A. Let B be a subring of A such that $y_1, \dots, y_m$ belong to B and generate $B \cap \mathfrak{m}$. Then in $\hat{A}$, every element of B, can be written as*

$$b = \sum_{\beta \in \mathbb{Z}^m_{\geq 0}} c_\beta y^\beta$$

where each c_β is either 0 or a unit of A contained in B.

Proof. Let $b \in B$. Since A is local, either $b \in A^\times$ (and we're done) or $b \in \mathfrak{m}$. In the second case, we can write b as a B-linear combination of $y_1, \dots, y_m$.

Suppose i is a positive integer, and we can write every $b \in B$ as a sum of an element b_i of the required form, and a B linear combination of degree i monomials in the elements $y_1, \dots, y_m$, so that

$$b = b_i + \sum \beta_f f(y_1, \dots, y_m)$$

where the sum runs over all f a degree i monomials, and each $\beta_f \in B$. Note each β_f is either in $A^\times$ or is in $\mathfrak{m}$, so can be written as $\beta_f = \sum \beta_{f,i} y_i$ where $\beta_{f,i} \in B$. This means that we can write b as the sum of some b_{i+1} of the form required, and a B linear combination of degree $i+1$ monomials in $y_1, \dots, y_m$ such that f_i and f_{i+1} have the same coefficients in degree $< i$.

Repeating this construction, we get an expansion for b in the required form. □

Proof of proposition 3.5. If w has these properties, then by taking $A = B = \hat{\mathcal{O}}_{\mathcal{X},\xi}$, we see that each $f = \sum f_\beta T^\beta$, and since each f_β is a unit, this tells us that $w(f) = \sum_{\beta \in \mathbb{Z}^m} \beta_i \alpha_i$. Moreover, if π is a uniformiser for $\mathfrak{o}$, we can write $\pi = u \prod_{i=1}^r T_i^{N_i}$ (by definition of our T_i), so that $w(\pi) = 1$, and so w extends v.

We first need to show that w is well defined.

Let $A = \hat{\mathcal{O}}_{\mathcal{X},\xi}$, and let $\mathfrak{m}$,κ its maximal ideal and residue field. Then to every expansion of the form in the previous lemma, we can associate a newton polyhedron Γ, which is the convex hull of the set

$$\{\beta \in \mathbb{Z}^r_{\geq 0} : c_\beta \neq 0\} + \mathbb{R}^r_{\geq 0} \subseteq \mathbb{R}^r.$$

Let Γ_c denote the set of points in $\mathbb{Z}^r_{\geq 0}$ that lie on a compact face of Γ, and let

$$b_c := \sum_{\beta \in \Gamma^c} \overline{c_\beta} z^\beta$$

where $\overline{c_\beta}$ denotes its image in the residue field.

Therefore, b_c is an element of $\kappa[z_1, \dots, z_r]$. We claim that it depends only on b, not on the way in which we write this expansion.

Let $b = \sum_{\beta \in \mathbb{Z}^r} c'_\beta z^\beta$ be another admissible expansion of b, with associated set Γ'_c and b'_c. Then

$$\sum_\beta (c_\beta - c'_\beta) z^\beta = 0.$$

Choosing admissible expansions for $c_\beta - c'_\beta$ that don't lie in $A^\times \cup \{0\}$ means we can write this as an expansion

$$0 = \sum_\beta d_\beta z^\beta$$

where each $\overline{d_\beta} = \overline{c_\beta} - \overline{c'_\beta}$ for all $\beta \in \Gamma_c \cup \Gamma'_c$.

Note that $\bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}$ is isomorphic to the polynomial rin gover κ in the residue classes of $z_1, \dots, z_2$ modulo $\mathfrak{m}^2$. Therefore, we see that each d_β must be 0, since $\sum d_\beta z^\beta = 0$ in some polynomial ring. Therefore, $\Gamma_c = \Gamma'_c$ and $b_c = b'_c$, so w doesn't depend on our choice of admissible expansion.

Now, let $m = \min\{\alpha \cdot \beta : \beta \in \Gamma_c : \overline{c_\beta} \neq 0\}$ (where $\alpha = (\alpha_1, \dots, \alpha_r)$, and $\cdot$ denotes the dot product in $\mathbb{R}^n$), and let Γ_c^α the points $\beta \in \Gamma_c$ such that $\alpha\beta = m$.

Let

$$f_\alpha = \sum_{\beta \in \Gamma_c^\alpha} \overline{c}_\beta z^\beta \in \kappa[z_1, \dots, z_r].$$

Then we see that m and f_α are entirely determined by f_c and Γ_c, so don't depend on our admissible expansion for f. Moreover, we see that

$$v_\alpha(b) = \min\{\alpha\beta : \alpha \in \Gamma_c, c_\beta \neq 0\} = m.$$

We now claim that v_α is a valuation. If f, g are non zero elements then $v_\alpha(+g) \geq \min\{v_\alpha(f), v_\alpha(g)\}$, and moreover $v_\alpha(fg) = v_\alpha(f) + v_\alpha(g)$, since $(fg)_\alpha = f_\alpha g_\alpha$.

Moreover, since $z_1, \dots, z_r$ are determined (up to a unit) in $\mathcal{O}_{\mathcal{X},\xi}$, it's easy to see $v_\alpha(f)$ doesn't depend on our choice $z_1, \dots, z_r$. Moreover, since $\pi = uz_1^{N_1} \dots z_r^{N_r}$, we get $v_\alpha(\pi) = \sum_{i=1}^r \alpha_i N_i = 1$, so v_α extends the p-adic valuation. □

Proposition 3.9. *Let x be a monomial point of X^{an}. Then TFAE:*

1. *The point x is divisorial.*
2. *The valuation v_x is discrete.*
3. *The space X^{an} has rational rank 1 at x*

Proof. We clearly have $1 \Rightarrow 2 \Rightarrow 3$, so let $x \in X^{an}$ have rational rank 1. Let $(\mathcal{X}, (E_i), \xi)$ and $\alpha \in \mathbb{R}^r_{\geq 0}$ be the data representing x. Since X^{an} has rational rank 1, we can assume that $\alpha \in \mathbb{Q}^r$.

Permuting the indices, let α_1 be the minimal index. Consider the blow up $h : \mathcal{X}' \to \mathcal{X}$ along the closure of ξ, and let E'_i denote the strict transform of E_i for all $i \neq 1$. Let E'_1 denote the exceptional divisor of the blow up. Let ξ' be the generic point of $E'_1 \cap \dots E'_r$. Then a relatively straightforward computation shows us that

$$(\mathcal{X}', (E'_i), \xi'), \alpha' = (\alpha_1, \alpha_2 - \alpha_1, \dots, \alpha_r - \alpha_1)$$

is monomial data for the point x. Moreover, we can clearly eliminate the zero entries in α' and their associated E'_i component.

Since the rational rank at x is equal to 1, all the α_i are integer multiplies of a common rational number $q > 0$. An induction of a basic argument tells us that after a finite number of steps, all of the α_i are 0 except for 1. Therefore, repeating this finitely many times, we get a monomial presentation with $r = 1$, ie, a divisorial presentation for x. □

4 Skeleta

Definition 4.1. Let $\mathcal{X}$ be an sncd model of X. Then the *Berkovich skeleton* associated to $\mathcal{X}$ is the set of monomial points that are associated to $(\mathcal{X}, E_1, \dots, E_r, \alpha, \xi)$ for some $E_1, \dots, E_r, \alpha, \xi$.

The big theorem about skeleta is the following.

Theorem 4.2. *Suppose X is proper and $\mathcal{X}$ is a proper sncd model, the inclusion* $\mathrm{Sk}(\mathcal{X}) \to X$ *is a homotopy equivalence.*

We won't prove this theorem, as it involves a lot of technical and awkward birational geometry, but we will characterise $\mathrm{Sk}(\mathcal{X})$ more easily.

Definition 4.3. Let $\mathcal{X}$ be an sncd model for X. Write $\mathcal{X}_{\mathbb{F}} = \sum_{i \in I} N_i E_i$, where E_i are the connected components of $\mathcal{X}_{\mathbb{F}}$. For J a non empty subset of I, let $E_J = \bigcup_{i \in J} E_i$.

Then the *dual complex* of $\mathcal{X}_k$ is a "simplicial complex" whose simplicies of dimension d are given by $\pi_0(E_J)$, where J runs through the subsets of I of cardinality $d+1$, glued together in the obvious way. NB: This is a "simplicial complex" rather than strictly a simplicial complex as we allow multiple edges between two vertices, however, since we're only concerned with the topological realisation, this is somewhat easy.

Explicitly: if J, J' are non empty subsets of I, and C, C' are connected components of J, J' respectively, then the simplex corresponding to C is a face of the simplex corresponding to C' if and only if $C \subseteq C'$. We'll also write ν_i to mean the vertex of $\Delta(\mathcal{X}_k)$.

We'll write $\Delta(\mathcal{X}_k)$ to mean this, and we'll call it the dual complex. Write $|\Delta(\mathcal{X}_k)|$ for its topological realisation.

In this way, the dual complex is a generalisation of the dual graph.

There is a natural map $\Phi : |\Delta(\mathcal{X}_k)| \to \mathrm{Sk}(\mathcal{X})$. This map is defined by first sending ν_i to the monomial point associated with $(\mathcal{X}, E_i)$, and we extend this to a map on all of $|\Delta(\mathcal{X}_k)|$ by interpolating in the fashion below.

Let $y \in |\Delta(\mathcal{X}_k)|$. Then there exists a unique face, τ, such that y lies in the interior of τ. By construction, τ corresponds to C, a connected component of E_J. Let ξ_τ be the generic point of C.

The vertices of τ are the irreducible components, E_i with $i \in J$, so we can represent y by $\beta \in \mathbb{R}^J$, where each $\beta_i \geq 0$ and $\sum_{i \in J} \beta_i = 1$. Therefore, define $\Phi(y)$ as the monomial point associated with $(\mathcal{X}, (E_i)_{i\in J}, (\beta_i/N_i)_{i \in J}, \xi_\tau)$.

Lemma 4.4. *The map defined above, Φ, is a bijection.*

Proof. Let $x \in \mathrm{Sk}(\mathcal{X})$, so that x corresponds to a valuation v_x. Then $\mathrm{center}_{\mathcal{X}}(x)$ is a generic point ξ of E_J for some J. For each $i \in J$, choose a local equation $E_i = \{T_i = 0\}$ where $T_i \in \mathcal{O}_{\mathcal{X},\xi}$. Set $\alpha_i = v_x(T_i)$.

Then $\Phi^{-1}(x)$ lies in the interior of the face τ corresponding to the connected component of ξ in E_J, and its barycentric co-ordinates are equal to $(\alpha_i N_i)_{i \in J}$. □

Theorem 4.5. *The map defined above, Φ, is a homeomorphism.*

Proof. Note since $|\Delta(\mathcal{X}_k)|$ is a simplicial complex, it's compact, and the target is Hausdorff. Therefore, we only need to show that it's continuous (since it is already a bijection).

Let $\mathcal{U}$ be an open subscheme of $\mathcal{X}$, and let $\Delta(\mathcal{U}_k)$ be the simplicial set associated to $\mathcal{U}_k)_{red}$. Note then that $\mathcal{U}_k \to \mathcal{X}_k$ is an open immersion, and induces a closed embedding $|\Delta(\mathcal{U}_k)| \to |\Delta(\mathcal{X}_k)|$. Moreover, covering $\mathcal{X}$ by finitely many $\mathcal{U}$s means that $|\Delta(\mathcal{U}_k)|$s form a closed cover. Therefore, we can assume that $\mathcal{X}$ is affine.

By definition of the Berkovich topology on X^{an}, all we have to prove is that for every regular function on X, the map

$$\begin{aligned} |\Delta(\mathcal{X}_k)| &\to \mathbb{R}_{\geq 0} \\ x &\mapsto |f(\Phi(x))| \end{aligned}$$

is continuous. This follows by the fact that the valuations v_α from the proof of proposition 3.5 are continuous in α. □

Let X be proper, and let $x \in X^{an}$ with valuation v_x.. We want to come up with a map $X \to \mathrm{Sk}(\mathcal{X})$. Let J be the set of $i \in I$ such that $\mathrm{center}_{\mathcal{X}}(x) \in E_i$. Let C be the connected component of E_J containing x with generic point ξ. For $i \in J$, choose a local equation $T_i = 0$ for E_i in $\mathcal{X}$ at $\mathrm{center}(x)$, and let $\alpha_i = v_x(T_i)$ (where we think of T_i as the image of T_i inside the residue field at x). Let $\alpha_i = v_x(T_i)$. Then our retraction $\rho_{\mathcal{X}}(x)$ is the monomial point associated to

$$(\mathcal{X}, (E_i)_{i\in J}, (\alpha_i)_{i \in J}, \xi).$$

It has the property such that it is the unique point, y of $\mathrm{Sk}(\mathcal{X})$, such that $\mathrm{center}(x)$ is contained in the closure of $\mathrm{center}(y)$, and y gives the same valuation to each local defining equation of E_i as x does. Again, it's somewhat easy to verify that $\rho_{\mathcal{X}}$ is continuous.

It turns out that $\rho_{\mathcal{X}}$ is the "end product" of our strong deformation retract. Proving that this is a strong deformation retract is a bit much for this talk