Shimura data in terms of $B(G, \mathbb{R})$

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Chapter 1

Introduction

This master's thesis is initially motivated by the hope that the rich theory of Shimura varieties also exists in a non-archimedean setup. To be more precise we recall a part of the classic setup:

Definition. Let G be a reductive group over $\mathbb{R}$ and X a conjugacy class of homomorphisms $h : \mathbb{S} \to G$. We state the following axioms:

1. SV1: The adjoint representation on $\mathrm{Lie}(G)_{\mathbb{C}}$ is of type $(1, -1), (0, 0), (-1, 1)$.
2. SV2: $\mathrm{ad}(h(i))$ is a Cartan involution of G^{ad}.

Remark. From now on we will call a pair (G, X) a Shimura datum, if it fulfills SV1 and SV2. Note that the classical definition has an additional condition, which we will not consider.

On the non-archimedean side we find in [RV14, Definition 5.1.] the definition a local Shimura datum.

Definition 1.0.1. Let $F/\mathbb{Q}_p$ be a finite extension and G a reductive group over F. A local Shimura datum is a triple $(G, b, \{\mu\})$, where $b \in B(G, F)$ and μ is a geometric conjugacy class, such that

1. $\{\mu\}$ is minuscule
2. $b \in B(G, \{\mu\})$.

Kottwitz introduced the pointed set $B(G)$ defined in terms of a reductive group G over a finite extension $F/\mathbb{Q}_p$ in 1985 (cf. [Kot85]). In a recent paper the construction of $B(G)$ was generalized to a reductive group G over any global or local field F (cf. [Kot14]).

Thus one might replace the non-archimedean local field in the above definition of a local Shimura datum by $\mathbb{R}$. Obviously we would like to recover the classical definition of a Shimura datum. This leads to the following conjecture;

Conjecture 1. *Let G be a reductive group over $\mathbb{R}$ and X be a conjugacy class of homomorphisms $h : \mathbb{S} \to G$. There is a 1:1 correspondence:*

$$\left\{ \begin{matrix} (G, X) \\ \textit{fulfilling SV1 and SV2} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{a minuscule conjugacy class } \mu : \mathbb{G}_m \to G_{\mathbb{C}}, \\ b \in B(G, \mathbb{R})_{bsc}\ ,\ \kappa_G(b) = \mu\ + \textit{other conditions...} \end{matrix} \right\}$$

In this thesis we will define a map and explain how to possibly achieve such a bijection. An essential tool connecting the above notions is given by G-bundles on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$, the twistor $\mathbb{P}^1$, which one can think of as the real analogue of the Fargues-Fontaine curve. Indeed we will prove in Chapter 4 the following;

Theorem. *There exists an essentialy surjective functor*

$$\mathcal{E} : \mathrm{Isoc}_{\mathbb{R}} \to \mathrm{Bun}_{\widetilde{\mathbb{P}^1_{\mathbb{R}}}} .$$

This will classify vector bundles on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$ as direct sum of certain vector bundles denoted by $\mathcal{O}_{\widetilde{\mathbb{P}^1_{\mathbb{R}}}}(\lambda)$ for $\lambda \in \frac{1}{2}\mathbb{Z}$. Via Tannakian formalism we will be able to generalize this theorem in Chapter 5;

Theorem. *There exists a bijection*

$$B(G, \mathbb{R}) \cong H^1_{ét}(X, G).$$

Thus G-bundles up to isomorphism are exactly given by elements in $B(G, \mathbb{R})$. In Chapter 6 we extend this classification motivated by the following theorem;

Theorem. *[Sch18, Proposition 6.1] The category of $U(1)$-equivariant semistable vector bundles on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$ is equivalent to the category of pure $\mathbb{R}$-Hodge structures.*

to the following theorem;

Theorem. *There exists a 1:1 correspondence*

$$\{\textit{basic elements } \mathrm{Hom}^{\otimes}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}))\} \longleftrightarrow \{U(1)\textit{-equivariant semistable } G\textit{-bundles on } X\}.$$

Evidently the left hand side is connected to the classical notion of a Shimura datum. Namely such tensor exact functors are equivalent to morphisms $h : \mathbb{S} \to H$, where H in an inner form of G. In the last chapter we collect all the results to first define a map

$$\left\{ \begin{matrix} (G, X) \\ \text{fulfilling SV1} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{a minuscule conjugacy class } \mu : \mathbb{G}_m \to G_{\mathbb{C}}, \\ b \in B(G, \mathbb{R})_{bsc} \ , \ \kappa_G(b) = \mu \end{matrix} \right\} \tag{1.1}$$

which we conjecture to be injective, if we additionaly assume that SV2 if fulfilled. Via the flag variety $\mathrm{Fl}_{G,\mu}(\mathbb{C})$ we will construct certain semistable $U(1)$-equivariant bundles as modifications of the trivial bundle at $\infty \in \widetilde{\mathbb{P}^1_{\mathbb{R}}}$ in the sense of Beauville-Laszlo. Thus we can compare both notions via semistable G-bundles

$$\begin{array}{ccc} & \text{semistable } G\text{-bundles on } \widetilde{\mathbb{P}^1_{\mathbb{R}}} & \\ \nearrow & & \nwarrow \\ \mathrm{Fl}_{G,\mu}(\mathbb{C}) & & B(G, \mathbb{R})_{bsc}. \end{array}$$

This will allow us to have a more precise idea of how the inverse of the map (1.1) above should look like.

Notation

We adopt the notation from [Mil17] and [DM18]. Furthermore algebraic groups will always be assumed to be over $\mathbb{C}$ unless specified otherwise, e.g. $\mathbb{G}_m$ always means $\mathbb{G}_{m,\mathbb{C}}$.

Throughout this thesis we will use X and $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$ interchangeably to denote the scheme

$$\mathrm{Proj}(\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2)).$$

Chapter 2

Preliminaries

2.1 Basic properties of $\mathrm{Isoc}_{\mathbb{R}}$

Throughout the chapter we set $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We begin by introducing some of the main constructions that will be needed.

Definition 2.1.1. A real isocrystal is a finite dimensional $\mathbb{C}$ vector space V with a grading $V = \bigoplus_{n\in\mathbb{Z}} V_n$ and an $\mathbb{R}$-linear automorphism $\phi : V \to V$ such that:

1. ϕ preserves the grading, i.e. $\phi(V_n) \subset V_n$
2. ϕ is antiholomorphic, i.e. $\phi(\lambda v) = \bar{\lambda}\phi(v)$ for $\lambda \in \mathbb{C}$, $v \in V$
3. $\phi^2|_{V_n} = (-1)^n$.

An isocrystal V is called pure (or isocline) of degree n, if it is concentrated in one degree, i.e. $V = V_n$.

Remark 2.1.2. For the rest of the thesis the term isocrystal will always refer to real isocrystals, unless stated otherwise. Isocrystals will usually be denoted by (V, α), if the grading on V is obvious from context. Otherwise, we may write (V, g, α), where $g : \mathbb{G}_m \to \mathrm{GL}(V)$ is the morphism corresponding to the induced grading or $(V, \bigoplus_{n\in\mathbb{Z}} V_n, \alpha)$ also indicating the grading.

Definition 2.1.3. Define the category of isocrystals $\mathrm{Isoc}_{\mathbb{R}}$ as follows:

- The objects are isocrystals as in the above definition
- A homomorphism between two isocrystals (V, α_V), (W, α_W) is a $\mathbb{C}$-linear morphism $f : V \to W$ which preserves the grading and commutes with the respective antiholomorphic automorphisms on V and W.

Lemma 2.1.4. *The full subcategory of pure isocrystals of degree n is equivalent to the category*

- *of $\mathbb{R}$-vector spaces, if n is even*
- *of $\mathbb{H}$-vector spaces, if n is odd.*

Proof. The even case is covered by descent of vector spaces, see e.g. [Bor12, Chapter AG §14]

Now assume that we are given a pure isocrystal of odd degree (V_n, α). We define the following structure on V_n:

$$l.v := i \cdot v,\ j.v := \alpha(v),\ k.v := i \cdot \alpha(v).$$

It is easy to check that the compatibility conditions in Definition 2.1.1. imply that the above structure indeed makes (V_n, l, j, k) an $\mathbb{H}$-vector space. Conversely an $\mathbb{H}$-vector space (W, l, j, k) consists of an $\mathbb{R}$-vector space W with three $\mathbb{R}$-linear automorphisms l, j, k. We make W into a complex vector space by defining $i.w = l(w)$. One then checks that (W, j) is an isocrystal, which follows from the compatibility conditions of a quaterionic structure. Finally one checks that the constructions preserve the homomorphisms. □

Later on we will not only need to consider isocrystals, but even $U(1)$-equivariant isocrystals. We give an ad-hoc definition, which will be motivated in Chapter 6.

Definition 2.1.5. Given an isocrystal $(V, \bigoplus_{n\in\mathbb{Z}} V_n, \phi)$, define the map

$$\phi \otimes \sigma_V : V \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}] \to V \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}]$$

as the direct sum of

$$\begin{aligned}(\phi \otimes \sigma_V)_n : V_n \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}] &\to V_n \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}] \\ v \otimes \lambda T^h &\mapsto \phi(v) \otimes \bar{\lambda} T^{n-h}.\end{aligned}$$

Definition 2.1.6. Define the category of $U(1)$-equivariant isocrystals $\mathrm{Isoc}_{\mathbb{R}}^{U(1)}$ as follows:

- The objects are isocrystals $(V, \phi) \in \mathrm{Ob}(\mathrm{Isoc}_{\mathbb{R}})$ with a comodule map $\gamma_V : V \to V \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}]$ respecting the grading, i.e. $\gamma_V(V_n) \subset V_n \otimes \mathbb{C}[T, T^{-1}]$, such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\gamma_V} & V \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}] \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi\otimes\sigma_V} \\ V & \xrightarrow{\gamma_V} & V \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}]. \end{array}$$

- A homomorphism between two objects is a homomorphism $f \in \mathrm{Hom}_{\mathrm{Isoc}_{\mathbb{R}}}(V, W)$, such that

$$\begin{array}{ccc} V & \xrightarrow{\gamma_V} & V \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\otimes id} \\ W & \xrightarrow{\gamma_W} & W \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}] \end{array}$$

  commutes.

Proposition 2.1.7. Equivalently, one can define the category $\mathrm{Isoc}_{\mathbb{R}}^{U(1)}$ as follows:

- The objects are isocrystals $(V, \phi) \in \mathrm{Ob}(\mathrm{Isoc}_{\mathbb{R}})$, such that for every $n \in \mathbb{Z}$ there is a decomposition into complex subspaces

$$V_n = \bigoplus_{m\in\mathbb{Z}} V_n^m, \quad \phi(V_n^m) \subset V_n^{n-m}.$$

- A homomorphism between two objects is a homomorphism $f \in \mathrm{Hom}_{\mathrm{Isoc}_{\mathbb{R}}}(V, W)$, such that f also preserves the above decomposition.

The proof is rather technical, but not difficult and relies on the fact that a comodule map

$$\gamma_V : V \to V \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}]$$

is equivalent to a decomposition

$$V = \bigoplus_{m \in \mathbb{Z}} V^m \ .$$

One then checks that the additional conditions in Definition 2.1.6 translate to the conditions stated in the lemma.

Remark 2.1.8. We will use both versions interchangeably. The second one has the advantage of being less technical, while the first one is useful to define certain descent morphisms later on in chapter 6.

Theorem 2.1.9. *The categories* $\mathrm{Isoc}_{\mathbb{R}}^{U(1)}$ *and* $\mathrm{Rep}_{\mathbb{R}}(\mathbb{S})$ *are (tensor) equivalent.*

Remark 2.1.10. Tensor equivalence will follow implicitly, since all our constructions will preserve tensor products.

We will prove this theorem by constructing explicit quasi-inverses.

Construction 2.1.11. Let $(V, V_{\mathbb{C}} = \bigoplus_{(p,q) \in \mathbb{Z}^2} V^{p,q})$ be a real Hodge structure with induced complex conjugation $\sigma : V_{\mathbb{C}} \to V_{\mathbb{C}}$. We will explain how to endow the vector space $V_{\mathbb{C}}$ with the structure of an $U(1)$-equivariant isocrystal as follows.

- Define the grading by $V_{\mathbb{C},n} = \bigoplus_{p+q=n} V^{p,q}$.
- To define the semilinear automorphism ϕ, we introduce the following map

  $$\begin{pmatrix} 1 \\ -1 \end{pmatrix} : V_C \to V_C \ , \ v^{p,q} \to (-1)^q v^{p,q}.$$

  Now define

  $$\phi := \begin{pmatrix} 1 \\ -1 \end{pmatrix} \circ \sigma$$

- The comodule map on the homogeneous elements $v^{p,q} \in V^{p,q}$ is given by

  $$\begin{aligned} \gamma_V : V_{\mathbb{C}} &\to V_{\mathbb{C}} \otimes \mathbb{C}[T, T^{-1}] \\ v^{p,q} &\to v^{p,q} \otimes T^q. \end{aligned}$$

Lemma 2.1.12. *The above construction yields a functor*

$$\mathcal{G} : \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}) \to \mathrm{Isoc}_{\mathbb{R}}^{U(1)} \ .$$

Proof. To see that the above construction is well-defined, i.e. that we a get an $U(1)$-equivariant isocrystal, one compares the compatibility conditions. We will only list the most important conditions:

- The commutativity of the diagram in Definition 2.1.6 (or equivalently the condition in Proposition 2.1.7) corresponds to the Hodge Symmetry of our structure, as for $v \in V^{p,q}$

$$\deg(\phi(v)) = \deg(\sigma(v^{p,q})) = p = n - q.$$

- For $v = v^{p,q} \in V_{\mathbb{C},p+q}$ we have

$$\phi^2(v) = \phi((-1)^p \sigma(v)) = (-1)^{p+q} v.$$

Furthermore for $f \in \mathrm{Hom}_{\mathrm{Rep}_{\mathbb{R}}(\mathbb{S})}(V, W)$, one can check that $f_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ is a morphism in $\mathrm{Isoc}_{\mathbb{R}}^{U(1)}$ and thus the construction can be made functorially. □

Construction 2.1.13. Now assume that we are given an $U(1)$-equivariant isocrystal (V, ϕ) and a comodule map γ_V. This comes with a decomposition

$$V_n = \bigoplus_{m \in \mathbb{Z}} V_n^m, \qquad \phi(V_n^m) \subset V_n^{n-m}.$$

- The Hodge decompositon is given by the following subspaces

$$V^{p,q} := V_{p+q}^q.$$

- The complex conjugation is given by

$$\sigma := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \circ \phi.$$

Lemma 2.1.14. *The above construction yields a functor*

$$\mathcal{H} : \mathrm{Isoc}_{\mathbb{R}}^{U(1)} \to \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}).$$

Proof. We first verify that the construction is well-defined.

- First let us check that V is spanned by the subspaces $V^{p,q}$. It is enough to show that each V_n is spanned by these subspaces, which is true by our construction:

$$\bigoplus_{p+q=n} V^{p,q} = \bigoplus_{q \in \mathbb{Z}} V_n^q = V_n \tag{2.1}$$

- We have Hodge symmetry:

$$\sigma(V^{p,q}) = \sigma(V_{p+q}^q) = \phi(V_{p+q}^p) \subset V_{p+q}^q = V^{q,p}.$$

- Let us check that σ is a complex conjugation. Semilinearity of σ follows from the semilinearity of ϕ. For $v = v^{p,q} \in V^{p,q}$, we have

$$\sigma^2(v) = \sigma((-1)^p \phi(v)) = (-1)^p \cdot \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \circ \phi^2(v) = (-1)^{p+q+p} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}(v) = (-1)^{2(p+q)} v = v.$$

Again we can extend this construction to be functorial. One checks that any morphism f in $\mathrm{Isoc}_{\mathbb{R}}^{U(1)}$ commutes with the defined complex conjugations and thus descends to the real forms. Obviously f will also preserve the Hodge decomposition. □

Lemma 2.1.15. *The functors*

$$\mathcal{G} : \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}) \to \mathrm{Isoc}_{\mathbb{R}}^{U(1)}, \qquad \mathcal{H} : \mathrm{Isoc}_{\mathbb{R}}^{U(1)} \to \mathrm{Rep}_{\mathbb{R}}(\mathbb{S})$$

are quasi-inverses.

Proof. Let us start with a Hodge structure $(V, V_{\mathbb{C}} = \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q})$ with complex conjugation $\sigma : V_{\mathbb{C}} \to V_{\mathbb{C}}$.

The associated isocrystal $\mathcal{G}(V, V_{\mathbb{C}} = \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q})$ has the same underlying vector space $V_{\mathbb{C}}$ and

- the decomposition is given by the subspaces $V_{\mathbb{C},n} = \bigoplus_{p+q=n} V^{p,q}$
- the comodule structure is given by $V_{\mathbb{C},n} = \bigoplus_{m\in\mathbb{Z}} V_n^m$, where $V_n^m = V^{n-m,m}$
- the semilinear automorphism is given by $\phi = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \circ \sigma$.

Now by construction $\mathcal{H}(\mathcal{G}(V, V_{\mathbb{C}} = \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q}))$ has the same decomposition as our original Hodge structure.

Since

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \circ \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \mathrm{id}$$

we also get back our original complex conjugation

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \circ \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \circ \sigma = \sigma.$$

Conversely let us start with an $U(1)$-equivariant isocrystal (W, φ, γ) (here γ is the comodule map). Then $\mathcal{H}(W, \varphi, \gamma)$ has the same underlying complex vector space and

- the decomposition is given by $W^{p,q} = W_{p+q}^q$
- complex conjugation is given by $\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \circ \varphi$.

Applying $\mathcal{G}$ to this Hodge structure we get the complex vector space W and

- the pure isocrystal of slope n is given by $\bigoplus_{p+q=n} W_n^q = W_n$
- the semilinear automorphism is given by $\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \circ \varphi = \varphi$
- for the comodule structure the subspaces $W^{p,q} = W_{p+q}^q$ are the degree q piece.

Thus we retrieve the original $U(1)$-equivariant isocrystal.

In the same manner one shows that morphisms are preserved under these constructions. □

Chapter 3

Properties of $B(G, \mathbb{R})$

In this chapter we will review the formalism of the functor $B(G, F)$, where G is a reductive group over a local or global field F. Furthermore we will introduce two invariants of this set, the Newton point and the natural transformation $\kappa_G : B(G, \mathbb{R}) \to \pi_1(G)_\Gamma$. In the last section we will give a more Tannakian description of $B(G, \mathbb{R})$ as tensor functors.

3.1 Basic definitions

In this section we will define $B(G, \mathbb{R})$. Although there is a much more general definition, we will mainly restrict our attention to the real case.

Definition 3.1.1. Let K/F be a finite Galois extension. Denote by Γ its Galois group. Let X be a Γ-module and denote by D the corresponding group of multiplicative type. A Galois gerb for K/F, bound by D, is an extension of groups

$$1 \to D(K) \to \mathcal{E} \to \Gamma \to 1.$$

Remark 3.1.2. Note that $D(K)$ has two Galois actions. First of all it has the natural Galois extension as K-rational points of the algebraic group D over F and secondly it has an action coming from conjugation induced by the short exact sequence. We require these two actions to agree.

Definition 3.1.3. Given an reductive group G over F and a Galois gerb, an algebraic cocycle consists of the following data:

- A morphism $v \in \mathrm{Hom}_K(D_K, G_K)$
- A map $x : \mathcal{E} \to G(K)$ denoted as $w \mapsto x_w$

satisfying the following conditions

- x is a 1-cocycle, i.e. $x_{w_1 w_2} = x_{w_1} w_1(x_{w_2})$, where $\mathcal{E}$ acts on $G(K)$ via its image in Γ
- $x_d = v(d)$ for all $d \in D(K)$
- $\mathrm{ad}(x_w) \circ \sigma(v) = v$ whenever w maps to $\sigma \in \Gamma$.

The set $Z^1_{alg}(\mathcal{E}, G(K))$ admits an action by $G(K)$ as follows

$$g.(v, x) = (\mathrm{ad}(g) \circ v, w \mapsto g x_w w(g)^{-1}).$$

One checks that this is a well-defined algebraic 1-cocycle again. This leads to

Definition 3.1.4. Define the pointed set $H^1_{alg}(\mathcal{E}, G(K))$ as the quotient of $Z^1_{alg}(\mathcal{E}, G(K))$ by the action of $G(K)$. The basepoint is simply the trivial homomorphism and the trivial abstract 1-cocycle.

Now we are able to define the first invariant of $H^1_{alg}(\mathcal{E}, G(K))$.

Lemma 3.1.5. *The projection*

$$\begin{aligned} Z^1_{alg}(\mathcal{E}, G(K)) &\to \mathrm{Hom}_K(D_K, G_K) \\ (v, x) &\mapsto v \end{aligned}$$

induces a well-defined map

$$H^1_{alg}(\mathcal{E}, G(K)) \to \left[\mathrm{Hom}_K(D_K, G_K)/\operatorname{Int}(G(K))\right]^{\Gamma}$$

Proof. This is an immediate consequence of the third condition of Definition 3.1.3. □

Remark 3.1.6. The induced map is referred to as Newton map and the image of an element in $H^1_{alg}(\mathcal{E}, G(K))$ is referred to as its Newton point. Given $b \in H^1_{alg}(\mathcal{E}, G(K))$ it is also denoted as $[v_b]$.

We also have a morphism

$$\mathrm{Hom}_F(D_F, Z(G)) \hookrightarrow \left[\mathrm{Hom}_K(D_K, G_K)/\operatorname{Int}(G(K))\right]^{\Gamma}.$$

given by basechange.

Definition 3.1.7. An element in $H^1_{alg}(\mathcal{E}, G(K))$ is basic, if its Newton point lies in the image of the map

$$\mathrm{Hom}_F(D_F, Z(G)) \hookrightarrow \left[\mathrm{Hom}_K(D_K, G_K)/\operatorname{Int}(G(K))\right]^{\Gamma}.$$

Now we specify to the real case by introducing the following Galois gerb;

Definition 3.1.8. Define the Weil group $W_{\mathbb{C}/\mathbb{R}}$ as

$$W_{\mathbb{C}/\mathbb{R}} := \mathbb{C}^* \coprod \mathbb{C}^* j$$

where

$$j^2 = -1, \qquad \lambda j = j\bar{\lambda} \quad \forall \lambda \in \mathbb{C}^*.$$

Remark 3.1.9. The Weil group fits into a short exact sequence

$$1 \to \mathbb{C}^* \to W_{\mathbb{C}/\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

which sends j to the non-trivial element.

Definition 3.1.10. For a reductive group G over $\mathbb{R}$, define

$$B(G, \mathbb{R}) := H^1_{alg}(W_{\mathbb{C}/\mathbb{R}}, G(\mathbb{C})).$$

We can arrange the above description in a more clear way.

Lemma 3.1.11. *Giving an element in* $(v,x) \in Z^1_{alg}(W_{\mathbb{C}/\mathbb{R}}, G(\mathbb{C}))$ *is equivalent to giving a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m(\mathbb{C}) & \longrightarrow & W_{\mathbb{C}/\mathbb{R}} & \longrightarrow & \Gamma & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle b_2} & & \| & & \\ 0 & \longrightarrow & G(\mathbb{C}) & \longrightarrow & G(\mathbb{C}) \rtimes \Gamma & \longrightarrow & \Gamma & \longrightarrow & 0, \end{array}$$

where $b_1 \in \mathrm{Hom}_{\mathbb{C}}(D_{\mathbb{C}}, G_{\mathbb{C}})$ *and* b_2 *is a group homomorphism.*

Proof. Giving a morphism b_1 is evidently the same as giving a morphism v. The main point is that the 1-cocycle condition transitions to a group homomorphism condition, if we define

$$b_2(w) := (x_w, \sigma_w)$$

where σ_w is the image of w in Γ. Conversely we can define

$$x_w := \mathrm{pr}_1(b_2(w)).$$

Finally note that the third condition in Definition 3.1.3 is always fulfilled. Indeed it is true for the $\mathbb{C}$-rational points (cf. [Kot14, 2.3]), which are dense inside the above group scheme. □

Remark 3.1.12. From now on we will mainly adopt this point of view of commutative diagrams modulo the equivalence relation, when speaking of $B(G, \mathbb{R})$. Sometimes we will denote such elements as $[(b_1, b_2)]$.

Construction 3.1.13. The above construction can be made functorial. Given a homomorphism $f : G \to G'$ of reductive groups over $\mathbb{R}$, we can construct a map

$$f_* : Z^1_{alg}(W_{\mathbb{C}/\mathbb{R}}, G(\mathbb{C})) \to Z^1_{alg}(W_{\mathbb{C}/\mathbb{R}}, G'(\mathbb{C}))$$

by post composition, i.e. given an element $b = (b_1, b_2)$ we can define

$$f_*(b) = (f \circ b_1, (f \rtimes id) \circ b_2).$$

This map is compatible with the respective actions and thus we also have a map

$$f_* : B(G, \mathbb{R}) \to B(G', \mathbb{R}).$$

Let us now draw a connection $B(\mathrm{GL}(V), \mathbb{R})$ and $\mathrm{Isoc}_{\mathbb{R}}$.

Lemma 3.1.14. *Given a complex vector space* V *the group* $\mathrm{GL}(V) \rtimes \Gamma$ *is isomorphic to the group of semilinear automorphisms, i.e. morphisms* T *such that* $T(v+w) = T(v) + T(w)$ *and* $T(\lambda v) = \theta(\lambda)T(v)$ *for some* $\theta \in \Gamma$. *Under this isomorphism* (id, σ) *corresponds to the semilinear automorphism* $\sigma_a(\Sigma_{b\in B}\lambda_b b) := \Sigma_{b\in B}\sigma(\lambda_b)b$ *for a basis* B *of* V.

Proof. [GW13, Chapter III §3.6 Proposition 6.]. □

Construction 3.1.15. Let V be a real vector space, $G = \mathrm{GL}(V)$ and $b = (b_1, b_2) \in Z^1_{alg}(W, G(\mathbb{C}))$. Then we can construct an isocrystal as follows

- The grading is given by $b_1 : \mathbb{G}_m \to \mathrm{GL}(V_C)$,

- The semilinear automorphism is given by $b_2(j) \in \mathrm{GL}(V_{\mathbb{C}}) \rtimes \Gamma$ (cf. Lemma 3.1.14).

Note that the element $(\mathrm{id}, \sigma) \in \mathrm{GL}(V_{\mathbb{C}}) \rtimes \Gamma$ can be identified with the semilinear automorphism $\mathrm{id} \otimes \sigma : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$.

Lemma 3.1.16. *The above datum yields a well-defined isocrystal.*

Proof. Let us go through the conditions in Definition 2.1.1.:

1. The calculation for $v_n \in V_n$ and $\lambda \in \mathbb{C}^*$

   $$b_1(\lambda)b_2(j)(v_n) = b_2(\lambda \cdot j)(v_n) = b_2(j \cdot \bar{\lambda})(v_n) = b_2(j)b_1(\bar{\lambda})(v_n) = b_2(j)(\bar{\lambda}^n \cdot v_n) = \lambda^n \cdot b_2(j)(v_n)$$

   shows that $b_2(j)$ preserves V_n
2. $b_2(j)$ is antiholomorphic since $j \in W_{\mathbb{C}/\mathbb{R}}$ is mapped to $\sigma \in \Gamma$ and thus so is $b_2(j)$ under the respective projection
3. this is a straightforward consequence of

   $$b_2(j)^2 = b_2(-1) = b_1(-1).$$ □

Remark 3.1.17. Thus we can interpret elements in $B(\mathrm{GL}(V), \mathbb{R})$ as certain isomorphism classes of isocrystals. We will interchangeably use both identifications.

One can check that $b \in B(GL(V), \mathbb{R})$ is basic if and only if the corresponding isomorphism class of the isocrystal is pure.

3.2 Some group cohomology

We will need some facts about group cohomology to construct the natural transformation κ_G. For this section our setup will be as follows; Γ is a finite group, A is an abelian group and these fit in a short exact sequence

$$1 \longrightarrow A \longrightarrow E \longrightarrow \Gamma \longrightarrow 1.$$

Furthermore we have a triple (M, Y, ξ), where M and Y are Γ-modules, and $\xi : Y \to \mathrm{Hom}(A, M)$ a Γ-map. This induces a map $\xi^\Gamma : Y^\Gamma \to \mathrm{Hom}(A, M)^\Gamma = \mathrm{Hom}_\Gamma(A, M)$. From general theory of group cohomology [Kot14, Section 3] we get an exact sequence

$$0 \longrightarrow H^1(\Gamma, M) \xrightarrow{\text{inf}} H^1(E, M) \xrightarrow{\text{res}} \mathrm{Hom}_\Gamma(A, M) \xrightarrow{\text{trans}} H^2(\Gamma, M),$$

where the restriction map is induced by precomposition (this is well-defined, since the action of A on M is trivial).

Definition 3.2.1. Define $H^1_Y(E, M)$ to be the fiber product of the following diagram

$$\begin{array}{ccc} H^1_Y(E, M) & \xrightarrow{r} & Y^\Gamma \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \xi^\Gamma} \\ H^1(E, M) & \xrightarrow{res} & \mathrm{Hom}_\Gamma(A, M) \end{array}$$

Example 3.2.2. Let us draw a first connection with the prior section. For this consider the Galois gerb:

$$1 \to D(K) \to \mathcal{E} \to \Gamma \to 1$$

and the triple given by $\big(T(K), \mathrm{Hom}_K(D,T), \xi : \mathrm{Hom}_K(D,T) \to \mathrm{Hom}(D(K), T(K))\big)$, where T is a torus over F.

Proposition 3.2.3. Let T be a torus. Then

$$\begin{array}{ccc} H^1_{alg}(\mathcal{E}, T(K)) & \longrightarrow & \mathrm{Hom}_F(D,T) \\ \downarrow & & \downarrow \\ H^1(\mathcal{E}, T(K)) & \longrightarrow & \mathrm{Hom}(D(K), T(K)) \end{array}$$

is cartesian, i.e. $H^1_{alg}(\mathcal{E}, T(K)) \cong H^1_{\mathrm{Hom}_K(D,T)}(\mathcal{E}, T(K))$.

Proof. We begin by specifying the maps. The upper map is the Newton point as we have

$$\big(\mathrm{Hom}_K(D_K, T_K)/\,\mathrm{Int}(T(K))\big)^\Gamma = \big(\mathrm{Hom}_K(D_K, T_K)\big)^\Gamma = \mathrm{Hom}_F(D,T).$$

The left map is the projection to the second factor. Instead of proving the universal property, we will show that the following map is an isomorphism

$$\begin{aligned} H^1_{alg}(\mathcal{E}, T(K)) &\to \mathrm{Hom}_F(D,T) \times_{\mathrm{Hom}(D(K),T(K))} H^1(\mathcal{E}, T(K)) \\ [(v,x)] &\to (v, [x]) \end{aligned}$$

- For surjectivity consider such a given $(v, [x])$. Then there is an obvious choice for a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D(K) & \longrightarrow & \mathcal{E} & \longrightarrow & \Gamma & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle v(K)} & & \downarrow{\scriptstyle [x]} & & \| & & \\ 0 & \longrightarrow & T(K) & \longrightarrow & T(K) \rtimes \Gamma & \longrightarrow & \Gamma & \longrightarrow & 0 \end{array}$$

  and this is commutative since v and $[x]$ have the same image in $\mathrm{Hom}(D(K), T(K))$.

- For injectivity assume $[(v,x)]$ and $[(v',x')]$ have the same image. Then we immediately have $v = v'$. Furthermore $[x] = [x']$ implies that $x' = t.x$ for some $t \in T(K)$, in which case $(v', x') = t.(v, x)$. We stress that $T(K)$ acts trivially on the first component since $T(K)$ is abelian. □

Definition 3.2.4. Assume again the general setup. Choose any (set-theoretic) section $s : \Gamma \to E$ and denote its image by X. For $f \in Z^1(A, M)$ and $w \in E$ define

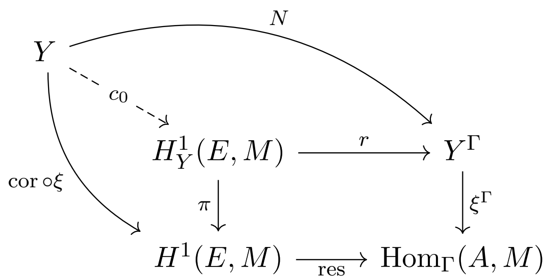
$$\mathrm{cor}(f)(w) = \sum_{x \in X} c_x \cdot f(c_x^{-1} w x)$$

where $c_x \in X$ is the unique representative, such that $wxA = c_xA$ or equivalently $c_x^{-1}wx \in A$. Then define the corestriction map

$$\mathrm{cor} : H^1(A, M) \to H^1(E, M)$$

as the induced map on the cohomology.

Definition 3.2.5. Define $c_0 : Y \to H^1_Y(E, M)$ via the following diagram



where $N : Y \to Y^\Gamma$ is the norm map.

Lemma 3.2.6. *The map $c_0 : Y \to H^1_Y(E, M)$ factors through Y_Γ.*

Proof. Obviously the norm map factors through Y_Γ. For $\mathrm{cor} \circ \xi$ this follows from [Kot14, Lemma 3.3.]. Thus we get again by the universal property an induced map $c : Y_\Gamma \to H^1_Y(E, M)$. □

Now we would like to use the group cohomology constructions to a more concrete case. For this we first give a definition of an abstract Tate-Nakayama triple.

Definition 3.2.7. Let Γ be a finite group, X and A Γ-modules and $\alpha \in H^2(\Gamma, \mathrm{Hom}(X, A))$. We say that (X, A, α) is a Tate-Nakayama triple, if for every subgroup Γ' of Γ:

- For all $r \in \mathbb{Z}$ cup product with $\mathrm{Res}_{\Gamma/\Gamma'}(\alpha)$ induces isomorphisms
$$H^r(\Gamma', X) \to H^{r+2}(\Gamma', A)$$
- $H^1(\Gamma', \mathrm{Hom}(X, A))$ is trivial.

Corresponding to $\alpha \in H^2(\Gamma, \mathrm{Hom}(X, A))$, we can choose an extension

$$1 \to \mathrm{Hom}(X, A) \to \mathcal{E} \to \Gamma \to 1. \tag{3.1}$$

Now let M be a Γ-module. Tensoring the canonical evaluation morphism

$$X \otimes \mathrm{Hom}(X, A) \to A$$

with M yields

$$M \otimes X \otimes \mathrm{Hom}(X, A) \to M \otimes A$$

and by the adjoint property we get

$$\xi : M \otimes X \to \mathrm{Hom}(\mathrm{Hom}(X, A), M \otimes A).$$

In terms of the setup at the beginning of this section $(M \otimes A, M \otimes X, \xi)$ is a triple relative to the short exact sequence (3.1). Thus we may apply the constructions made before to get a morphism

$$c : (M \otimes X)_\Gamma \to H^1_Y(\mathcal{E}, M \otimes A).$$

Lemma 3.2.8. *Assume that M is torsion free as an abelian group, then*

$$c : (M \otimes X)_\Gamma \to H^1_Y(\mathcal{E}, M \otimes A)$$

is an isomorphism.

Proof. [Kot14, Lemma 4.1.] □

3.3 Algebraic fundamental group and κ_G

Finally we turn to the case we are interested in. Choose

- $\Gamma = \mathrm{Gal}(\mathbb{C}, \mathbb{R})$
- $X = \mathbb{Z}$ with trivial Galois action
- $A = \mathbb{C}^*$ with the natural Galois action
- $\alpha \in H^2(\Gamma, \mathbb{C}^*)$ to be the fundamental class,

where α corresponds to the short exact sequence

$$1 \to \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^*) \to W_{\mathbb{C}/\mathbb{R}} \to \Gamma \to 1.$$

From local class field theory we get the following theorem;

Theorem 3.3.1. *The triple* $(\mathbb{Z}, \mathbb{C}^*, \alpha)$ *is a Tate-Nakayama triple for* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$*.*

For any torus T over $\mathbb{R}$ its cocharacter group $X_*(T)$ is a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ module. In the notation of our preceding section this will be our module M.

Lemma 3.3.2. *The evaluation map*

$$X_*(T) \otimes \mathbb{C}^* \to T(\mathbb{C})$$

is an isomorphism.

Proof. Choosing an explicit isomorphism $T_{\mathbb{C}} \cong \mathbb{G}_m^r$ the statement reduces to checking that

$$\mathbb{Z}^r \otimes \mathbb{C}^* \to (\mathbb{C}^*)^r$$

is an isomorphism, which is straightforward. □

Lemma 3.3.3. *The map*

$$c : X_*(T)_\Gamma \to H^1_{X_*(T)}(W, X_*(T) \otimes \mathbb{C}) \to H^1_{alg}(W, T(\mathbb{C}))$$

is an isomorphism.

Proof. The first map is an isomorphism by Lemma 3.2.8. The second map is an isomorphism by Lemma 3.3.2. and Proposition 3.2.3. □

Recall that by definition $B(T, \mathbb{R}) = H^1_{alg}(W, T(\mathbb{C}))$ and thus the above lemma gives us a map

$$c^{-1} : B(T, \mathbb{R}) \to X_*(T)_\Gamma.$$

Based on this we will define κ_G. The last notion that we need is that of the algebraic fundamental group.

Construction 3.3.4. Let G be a reductive group over $\mathbb{C}$. Denote its derived subgroup by G^{ss}. The universal cover of G^{ss} is denoted by G^{sc}. Thus we have a homomorphism

$$\rho : G^{sc} \to G^{ss} \hookrightarrow G.$$

Now choose a maximal torus $T \subset G$ and write $T^{sc} = \rho^{-1}(T)$, which is a maximal torus for G^{sc}. We can define the algebraic fundamental group in terms of T as

$$\pi_1(G, T) := X_*(T)/\rho_*(X_*(T^{sc})).$$

Now assume that T' is another maximal torus. Then we can find $g \in G(\mathbb{C})$ such that $T' = gTg^{-1}$. Conjugation by g then induces a map

$$g_* : \pi_1(G, T) \cong \pi_1(G, T').$$

This map can be shown to be a isomorphism and independent of the choice of g (cf. [Bor98, Lemma 1.2.]). Thus, once chosen a maximal torus, we will speak of the algebraic fundamental group $\pi_1(G)$.

Remark 3.3.5. Alternatively we could have defined

$$\pi_1(G) = X_*(T)/\langle \Phi^\vee \rangle,$$

where $\Phi^\vee$ are the coroots of G.

Definition 3.3.6. Given a reductive group G over $\mathbb{R}$ with a choice of a maximal torus $T \subset G$, define its algebraic fundamental group as

$$\pi_1(G) := \pi_1(G_{\mathbb{C}}).$$

Remark 3.3.7. In fact $\pi_1(G)$ admits a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action. If $T_{\mathbb{C}} \subset G_{\mathbb{C}}$ is a maximal torus, then $\sigma(T_{\mathbb{C}})$ is again a maximal torus and we can find $g \in G(\mathbb{C})$, such that $\sigma(T) = gTg^{-1}$. Then we can define the action of σ to be the following composition

$$\pi_1(G, T) \xrightarrow{\sigma_*} \pi_1(G, \sigma(T)) \xrightarrow{g_*} \pi_1(G, T)$$

which is again well-defined.

The following proposition motivates the name "algebraic fundamental group".

Proposition 3.3.8. [Bor98, Proposition 1.11.] Let K be either $\mathbb{R}$ or $\mathbb{C}$. For a connected reductive K-group G there is a canonical isomorphism

$$\pi_1(G) \xrightarrow{\sim} \mathrm{Hom}\left(\pi_1^{top}(\mathbb{G}_m(\mathbb{C})), \pi_1^{top}(\mathbb{G}(\mathbb{C}))\right)$$

where π_1^{top} is the usual topological fundamental group.

Now we will give the construction of κ_G. For more details we refer the reader to [Kot14, Section 9].

Construction 3.3.9. The construction is done in three steps

1. First assume that $G = T$ is a torus. By the prior discussion we have

$$X_*(T)_\Gamma = \pi_1(T)_\Gamma$$

   and thus by Lemma 3.3.3. a well-defined morphism

$$\kappa_T : X_*(T)_\Gamma \to B(T, \mathbb{R}).$$

2. Now assume that the derived subgroup of G is simply connected and denote the quotient of G by its derived subgroup as D. Then the following commutative diagram

$$\begin{array}{ccc} B(G,\mathbb{R}) & \xrightarrow{\kappa_G} & \pi_1(G)_\Gamma \\ \downarrow & & \| \\ B(D,\mathbb{R}) & \xrightarrow{\kappa_D} & \pi_1(D)_\Gamma \end{array}$$

forces us to define κ_G as the arrow, which makes the diagram commute.

3. The general case uses the existence of a z-extension, that is a short exact sequence over $\mathbb{R}$

$$1 \to Z \to G' \to G \to 1$$

such that

- Z is a central torus in G',
- Z is obtained by Weil restriction of scalars of a torus of $\mathbb{C}$,
- the derived subgroup of G' is simply connected.

One precedes to show that the maps.

$$B(G',\mathbb{R}) \to B(G,\mathbb{R}) \ , \ \pi_1(G')_\Gamma \to \pi_1(G)_\Gamma$$

are quotient maps by the actions of $B(Z,\mathbb{R})$ and $\pi_1(Z)_\Gamma$ and that $\kappa_{G'}$ is equivariant relative to these actions. Thus we get a commutative diagram

$$\begin{array}{ccc} B(G',\mathbb{R}) & \xrightarrow{\kappa_{G'}} & \pi_1(G')_\Gamma \\ \downarrow p & & \downarrow p \\ B(G,\mathbb{R}) & \xrightarrow{\kappa_G} & \pi_1(G)_\Gamma \end{array}$$

and there is a unique choice for κ_G.

3.4 $B(G,\mathbb{R})$ as G-isocrystals

This section is mainly inspired by [DOR10, Chapter IX], which proves the results below in the non-archimedean case.

Throughout this section let G be a reductive group over $\mathbb{R}$.

Definition 3.4.1. A G-isocrystal, also denoted as G-$\mathrm{Isoc}_\mathbb{R}$, is an exact tensor functor $N : \mathrm{Rep}_\mathbb{R}(G) \to \mathrm{Isoc}_\mathbb{R}$, such that

$$\omega_{iso} \circ N = \omega^G \otimes_\mathbb{R} \mathbb{C}.$$

The first aim of this section is to prove that

Theorem 3.4.2. *There exists a bijection between $B(G,\mathbb{R})$ and the set of isomorphism classes of G-$\mathrm{Isoc}_\mathbb{R}$.*

We are going to split this theorem into parts.

Construction 3.4.3. Let $b = (b_1, b_2)$ be an element in $Z^1_{alg}(W_{\mathbb{C}/\mathbb{R}}, G(\mathbb{C}))$. To this we can associate a G-$\mathrm{Isoc}_{\mathbb{R}}$ as follows. By the functoriality introduced in the last section, we get an element

$$r_*(b) \in Z^1_{alg}(W, GL(V_{\mathbb{C}})).$$

By Proposition 3.1.16 this yields an isocrystal. More concretely given a representation $(V, r) \in \mathrm{Rep}_{\mathbb{R}}(G)$, we can endow $V_{\mathbb{C}}$ with the grading induced by

$$\mathbb{G}_m \xrightarrow{b_1} G_{\mathbb{C}} \xrightarrow{r_{\mathbb{C}}} \mathrm{GL}(V_{\mathbb{C}}).$$

If $b_2(j) = (A, \sigma) \in G(\mathbb{C}) \rtimes \Gamma$, then the semilinear automorphism is given by

$$r_{\mathbb{C}}(A) \circ \sigma.$$

Proposition 3.4.4. There exists a well-defined map

$$\begin{aligned} N \colon Z^1_{alg}(W, G(\mathbb{C})) &\to G\text{-}\,\mathrm{Isoc}_{\mathbb{R}} \\ b = (b_1, b_2) &\mapsto \big(N_b : (V, r) \mapsto (V_{\mathbb{C}}, r_{\mathbb{C}} \circ b_1, r_{\mathbb{C}}(A) \circ \sigma)\big) \end{aligned}$$

Proof. We have checked that $N_b(V, r)$ is an isocrystal. If $f : (V, r) \to (V', r')$ is a morphism in $\mathrm{Rep}_{\mathbb{R}}(G)$, then $f_{\mathbb{C}} : V_{\mathbb{C}} \to V'_{\mathbb{C}}$ is compatible with the grading and the semilinear automorphism, since it commutes with $r_{\mathbb{C}}$. Furthermore it is straightforward, but mostly technical, to verify that N_b is a tensor exact functor. $\square$

Lemma 3.4.5. *The above map descends to a well-defined map*

$$\begin{aligned} N \colon B(G, \mathbb{R}) &\to G\text{-}\,\mathrm{Isoc}_{\mathbb{R}} /{\sim} \\ [b] = [(b_1, b_2)] &\mapsto [N_b : (V, r) \mapsto (V_{\mathbb{C}}, r_{\mathbb{C}} \circ b_1, r_{\mathbb{C}}(A) \circ \sigma)]. \end{aligned}$$

Proof. We have to check that $g.b = b'$ implies $N_b \cong N_{b'}$. We will check that the required natural isomorphism $\beta : N_b \to N_{g.b}$ is given by:

$$\begin{aligned} \beta(V, r) \colon (V_{\mathbb{C}}, r_{\mathbb{C}} \circ b_1, r_{\mathbb{C}}(A) \circ \sigma) &\to (V \otimes \mathbb{C}, r_{\mathbb{C}} \circ \mathrm{ad}(g) \circ b_1, r_{\mathbb{C}}(g \cdot A \cdot \sigma(g)^{-1}) \circ \sigma) \\ v &\mapsto r_{\mathbb{C}}(g)v. \end{aligned}$$

Let $v \in V^n$. Then

$$r_{\mathbb{C}}(g(b_1(\lambda)g^{-1})r_{\mathbb{C}}(g)v = \lambda^n r_{\mathbb{C}}(g)v$$

and thus the above morphism preserves the grading. We can conclude by the commutativity of the following diagram

$$\begin{array}{ccc} V_{\mathbb{C}} & \xrightarrow{r_{\mathbb{C}}(A)\circ\sigma} & V_{\mathbb{C}} \\ \downarrow{\scriptstyle r_{\mathbb{C}}(g)} & & \downarrow{\scriptstyle r_{\mathbb{C}}(g)} \\ V_{\mathbb{C}} & \xrightarrow{r_{\mathbb{C}}(g\cdot A\cdot\sigma(g)^{-1})\circ\sigma} & V_{\mathbb{C}} \end{array}$$

that $\beta(V, r)$ is indeed a morphism in $\mathrm{Isoc}_{\mathbb{R}}$. Thus β is a natural transformation. Its inverse is given by $v \mapsto r_{\mathbb{C}}(g)^{-1}v$ (which is similarly checked to be again a morphism in $\mathrm{Isoc}_{\mathbb{R}}$). $\square$

Lemma 3.4.6. *The map* $N \colon B(G, \mathbb{R}) \to G\text{-}\,\mathrm{Isoc}_{\mathbb{R}} /{\sim}$ *is surjective.*

Proof. Assume we have a G-$\mathrm{Isoc}_{\mathbb{R}}$, i.e. a tensor exact functor $N\colon \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Isoc}_{\mathbb{R}}$ such that $\omega_{iso} \circ N = \omega^G \otimes_{\mathbb{R}} \mathbb{C}$. Consider the following diagram

$$\begin{array}{ccccc} \mathrm{Rep}_{\mathbb{R}}(G) & \xrightarrow{N} & \mathrm{Isoc}_{\mathbb{R}} & \xrightarrow{\mathrm{grad}} & \mathrm{Rep}_{\mathbb{C}}(\mathbb{G}_m) \\ & {\scriptstyle \omega^G \otimes_{\mathbb{R}} \mathbb{C}} \searrow & \downarrow & \swarrow {\scriptstyle \omega^{\mathbb{G}_m}} & \\ & & \mathrm{Vect}_{\mathbb{C}} & & \end{array}$$

where $\mathrm{grad} : \mathrm{Isoc}_{\mathbb{R}} \to \mathrm{Rep}_{\mathbb{C}}(\mathbb{G}_m)$ forgets the semilinear automorphism and the vertical arrow also forgets the grading. Then the left triangle commutes by assumption and the right triangle always commutes. This induces a natural transformation

$$(N \circ gr)^* : \underline{\mathrm{Aut}}(\omega^{\mathbb{G}_m}) \to \underline{\mathrm{Aut}}(\omega^G \otimes_{\mathbb{R}} \mathbb{C})$$

between functors

$$\mathbb{C}\text{-alg} \to \mathrm{Grps}.$$

The latter functor is represented by $G_{\mathbb{C}}$ and thus by Yoneda we get a morphism

$$b_1 : \mathbb{G}_m \to G_{\mathbb{C}}.$$

By construction the grading on $N(V,r)$ is induced by $r_{\mathbb{C}} \circ b_1$.

For each representation denote $N(V,r) = (V_{\mathbb{C}}, \phi_r)$. We can define the following automorphism of the fiber functor

$$\alpha(V,r) : V_{\mathbb{C}} \xrightarrow{\phi_r \circ \sigma} V_{\mathbb{C}}.$$

By Tannaka duality such an automorphism is given by an element $A \in \mathbb{G}(\mathbb{C})$. Then we define $b_2(j) = (A, \sigma)$. Note that again by construction $\phi_r = r_{\mathbb{C}}(A) \circ \sigma$.

We claim that $b = (b_1, b_2)$ is an element in $Z^1_{alg}(W, G(\mathbb{C}))$. This comes down to checking that $b_2(\lambda j) = b_2(j\bar{\lambda})$. For each representation we have

$$r_{\mathbb{C}}(b_1(\lambda)A) \circ \sigma = r_{\mathbb{C}}(b_1(\lambda)) \circ \phi_r = \phi_r \circ r_{\mathbb{C}}(b_1(\lambda)) = r_{\mathbb{C}}(A) \circ r_{\mathbb{C}}(\sigma(b_1(\bar{\lambda}))) \circ \sigma = r_{\mathbb{C}}(A\sigma(b_1(\bar{\lambda}))) \circ \sigma.$$

The second equality follows since by definition $\phi_r = r_{\mathbb{C}}(A) \circ \sigma$ preserves the grading and the third equality follows from $\sigma \circ M = \sigma(M) \circ \sigma$, which holds for any automorphism. But since this equality holds for any $(V,r) \in \mathrm{Rep}_{\mathbb{R}}(G)$, we can conclude that

$$b_1(\lambda)A = A\sigma(b_1(\bar{\lambda})).$$

This implies the desired equality.

Finally we have that $N_b \cong N$. The isomorphism is a consequence of our construction. In particular it follows from the following equalities

$$gr \circ N = gr \circ N_b \ , \ \phi_r = r_{\mathbb{C}}(A) \circ \sigma. \qquad \square$$

Lemma 3.4.7. *The map* $N\colon B(G,\mathbb{R}) \to G\text{-}\mathrm{Isoc}_{\mathbb{R}}/\!\sim$ *is injective.*

Proof. Assume that $\alpha : N_b \cong N_{b'}$. Then we get an induced automorphism of the fiber functor $\omega^G \otimes \mathbb{C}$ given by post composing N_b (resp. $N_{b'}$) with ω_{iso}. This is equivalent to an element $g \in G(\mathbb{C})$, which implies that

$$N_b(V,r) \xrightarrow{r_{\mathbb{C}}(g)} N_{b'}(V,r)$$

is an isomorphism in $\mathrm{Isoc}_{\mathbb{R}}$. In particular $v \in V_n'$ if and only if $v \in r_{\mathbb{C}}(g)V_n$, i.e.

$$\mathrm{grad} \circ N_{b'} = \mathrm{grad} \circ N_b \circ \mathrm{ad}(g)$$

and we conclude that

$$b_1' = ad(g)b_1.$$

Finally the commutative diagram

$$\begin{array}{ccc} V_{\mathbb{C}} & \xrightarrow{r_{\mathbb{C}}(A)\circ\sigma} & V_{\mathbb{C}} \\ \downarrow{\scriptstyle r_{\mathbb{C}}(g)} & & \downarrow{\scriptstyle r_{\mathbb{C}}(g)} \\ V_{\mathbb{C}} & \xrightarrow{r_{\mathbb{C}}(A')\circ\sigma} & V_{\mathbb{C}} \end{array}$$

implies that

$$r_{\mathbb{C}}(A') = r_{\mathbb{C}}(g \cdot A \cdot \sigma(g)^{-1})$$

for all representations and thus

$$A' = g \cdot A \cdot \sigma(g)^{-1}.$$

We have shown that $b' = g.b$, i.e. $[b] = [b']$ and thus N is injective. □

Definition 3.4.8. Define $\mathrm{Hom}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Isoc}_{\mathbb{R}})$ to be the set of isomorphism classes of exact tensor functors.

Lemma 3.4.9. *There is an isomorphism between* $G\text{-}\mathrm{Isoc}_{\mathbb{R}}/\!\sim$ *and* $\mathrm{Hom}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Isoc}_{\mathbb{R}})$*.*

Proof. As there is an obvious inclusion $G\text{-}\mathrm{Isoc}_{\mathbb{R}}/\!\sim\, \to \mathrm{Hom}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Isoc}_{\mathbb{R}})$, we only need to verify surjectivity. Given such an element

$$N \in \mathrm{Hom}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Isoc}_{\mathbb{R}}),$$

the composition with ω_{iso} is necessarily isomorphic to $\omega^G \otimes \mathbb{C}$: There exists by [DM18, Theorem 3.2.] an isomorphism between tensor functors

$$\omega' : \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Vect}_{\mathbb{C}}$$

and G torsors. But $H^1(\mathbb{C}, G) = *$ implies that all G-torsors are trivial and thus all fiber functors are isomorphic to $\omega^G \otimes \mathbb{C}$. □

Corollary 3.4.10. *There exists a bijection*

$$B(G, \mathbb{R}) \cong \mathrm{Hom}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Isoc}_{\mathbb{R}}).$$

Proof. Combine Lemma 3.4.9. and Theorem 3.4.2. □

Chapter 4

Vector bundles on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$

In this chapter we devote our attention to the classification of vector bundles on $X := \widetilde{\mathbb{P}^1_{\mathbb{R}}}$, the twistor $\mathbb{P}^1$. Our main result will yield an essentially surjective functor $\mathcal{E} : \mathrm{Isoc}_{\mathbb{R}} \to \mathrm{Bun}_X$.

4.1 Semistable vector bundles

The complex projective space $\mathbb{P}^1_{\mathbb{C}}$ has two real forms corresponding to the elements of the Brauer group $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. The trivial form is the real projective space $\mathbb{P}^1_{\mathbb{R}}$ and the non-trivial form gives the twistor $\mathbb{P}^1$. An explicit realization is given by

$$\widetilde{\mathbb{P}^1_{\mathbb{R}}} := \mathrm{Proj}(\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2))$$

as the following lemma will show.

Lemma 4.1.1. *The automorphism $f(z) = -\frac{1}{\bar{z}}$ of $\mathbb{P}^1_{\mathbb{C}}$ is a descent datum for the Galois covering $p : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ and the descent is given by $\mathrm{Proj}(\mathbb{R}[x, y, z]/I)$ for $I = (x^2 + y^2 + z^2)$.*

Proof. As $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \times \mathrm{Spec}(\mathbb{C}) \xrightarrow{\sim} \mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$, a descent datum for a scheme Z is equivalent to an action $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \times Z \to Z$, such that the following diagram commutes:

$$\begin{array}{ccc} Z & \xrightarrow{\sigma_Z} & Z \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{\sigma} & \mathbb{C} \end{array}$$

for all $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. In our case we have $Z = \mathbb{P}^1_{\mathbb{C}}$ and we choose $\sigma_Z = f$.

Then consider the morphism $\phi\colon \mathbb{C}[x, y, z]/I \to \mathbb{C}[a, b]$

$$x \mapsto \frac{a^2 + b^2}{2}, \qquad y \mapsto \frac{i(a^2 - b^2)}{2}, \qquad z \mapsto i(ab)$$

which induces an isomorphism between $\mathbb{P}^1_{\mathbb{C}}$ and the conic $V(I) \subset \mathbb{P}^2_{\mathbb{C}}$.

This isomorphism induces an isomorphism of descent data by defining the involution $\sigma_1 = \phi^{-1} \circ f \circ \phi$ on $\mathbb{C}[x, y, z]/I$. Explicit calculation shows that σ_1 is the ordinary complex conjugation on $\mathbb{C}[x,y,z]/I$, thus the real form is $\mathbb{R}[x,y,z]/I$. Putting everything together we get a map $\pi : \mathbb{P}^1_{\mathbb{C}} \to \widetilde{\mathbb{P}^1_{\mathbb{R}}}$ and $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$ is isomorphic to the descent of $\mathbb{P}^1_{\mathbb{C}}$ via f since the respective descent data are. □

Lemma 4.1.2. *The canonical divisor ω_X is isomorphic to $\mathcal{O}_X(-1)$.*

Proof. We consider X as subvariety of $\mathbb{P}^2_{\mathbb{R}}$. We have that $\omega_{\mathbb{P}^2_{\mathbb{R}}} \cong \mathcal{O}_{\mathbb{P}^2_{\mathbb{R}}}(-3)$. Now adjunction formula tells us that

$$\omega_X \cong \omega_{\mathbb{P}^2_{\mathbb{R}}} \otimes \mathcal{O}_{\mathbb{P}^2_{\mathbb{R}}}((X)) \otimes \mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^2_{\mathbb{R}}}(-3+2) \otimes \mathcal{O}_X \cong \mathcal{O}_X(-1).$$ □

Definition 4.1.3. For $\lambda \in \frac{1}{2}\mathbb{Z}$ define the following vector bundles:

$$\mathcal{O}_X(\lambda) = \begin{cases} \mathcal{O}_X(\lambda) & \text{for } \lambda \in \mathbb{Z} \\ \pi_* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda) & \text{for } \lambda \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \end{cases}$$

Remark 4.1.4. For the integers we get back the classical twisted line bundles $\mathcal{O}_X(\lambda)$ whereas for proper fractions $\frac{\lambda}{2}$ we get vector bundles of rank 2 as π is a degree 2 covering.

Lemma 4.1.5. *The cohomology of $\mathcal{O}_X(\lambda)$ is as follows:*

$$H^0(X, \mathcal{O}_X(\lambda)) = \begin{cases} \mathbb{R}^{2\lambda+1} & \text{for } \lambda \in (\frac{1}{2}\mathbb{Z})_{\geq 0} \\ 0 & \text{else} \end{cases}$$

$$H^1(X, \mathcal{O}_X(\lambda)) = \begin{cases} \mathbb{R}^{-(2\lambda+1)} & \text{for } \lambda \in \frac{1}{2}\mathbb{Z}_{\leq -1} \\ 0 & \text{else} \end{cases}$$

Proof. If $\lambda \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then we conclude $H^i(X, \mathcal{O}_X(\lambda)) \cong H^i(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda))$ as $\pi : \mathbb{P}^1_{\mathbb{C}} \to X$ is affine. Thus this case is covered by the computation of cohomology on $\mathbb{P}^1_{\mathbb{C}}$. For the line bundles $\mathcal{O}_X(\lambda)$, where now $\lambda \in \mathbb{Z}$, we compute the global sections and then conclude via Serre duality for the first cohomology groups. By [Har77, Exercise 5.14] and [Har77, Exercise 6.4] we have an isomorphism of graded rings

$$R = \mathbb{R}[x,y,z]/(x^2+y^2+z^2) \cong \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)).$$

One then checks that the generators of R_n as $\mathbb{R}$-vector space are given by

$$\{\, y^b z^c \mid b+c=n \,\} \coprod \{\, xy^b z^c \mid b+c=n-1 \,\}.$$

Thus $\dim_{\mathbb{R}}(H^0(X, \mathcal{O}_X(n))) = \dim_{\mathbb{R}}(R_n) = (n+1)+n = 2n+1$ as desired. By Serre duality we also get the first cohomology groups as claimed above. □

Definition 4.1.6. A vector bundle $\mathcal{V}$ on X (resp. $\mathbb{P}^1_{\mathbb{C}}$) is called pure, if $\mathcal{V} \cong \mathcal{O}_X(\frac{\lambda}{2})^{\oplus n}$ (resp. $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\lambda)$) for some $\lambda \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Lemma 4.1.7. *With the above notation we have:*

$$\pi^* \mathcal{O}_X(\lambda) \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda) & \textit{for } \lambda \in \mathbb{Z} \\ \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda) & \textit{for } \lambda \in (\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}) \end{cases}$$

Proof.
- If $\lambda \in \mathbb{Z}$, then $\pi^*\mathcal{O}_X(\lambda)$ is a line bundle. Using finite locally free base change [Sta18, Tag 0CKW], we can compute the cohomology

$$H^*(\mathbb{P}^1_{\mathbb{C}}, \pi^*\mathcal{O}_X(\lambda)) \cong H^*(X, \mathcal{O}_X(\lambda)) \otimes_{\mathbb{R}} \mathbb{C}.$$

Since a line bundle on $\mathbb{P}^1_{\mathbb{C}}$ is determined by its cohomology, we can conclude with Lemma 4.1.5.

- For $\lambda \in (\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z})$ this is [FF18, Exemple 5.6.18.] □

We recall the Harder-Narasimhan formalism. Let $\mathcal{V}$ be a vector bundle on X. Then we define the degree of $\mathcal{V}$ as $\deg(\mathcal{V}) := \deg(\det(\mathcal{V}))$ and its slope as $\mu(\mathcal{V}) = \frac{\deg(\mathcal{V})}{rk(\mathcal{V})}$.

Definition 4.1.8. Let Z be a scheme and $\mathcal{V}$ a locally free sheaf on Z. A locally free subsheaf $\mathcal{W} \subset \mathcal{V}$ is a subbundle, if it is Zariski locally a direct summand of $\mathcal{V}$.

Definition 4.1.9. Let $\mathcal{V}$ be a vector bundle on a scheme Z. Then $\mathcal{V}$ is called semistable, if

$$\mathcal{W} \text{ is a subbundle of } \mathcal{V} \implies \mu(\mathcal{W}) \leq \mu(\mathcal{V}).$$

It is called stable, if the above inequality is always strict for proper subbundles.

Definition 4.1.10. Let $\mathcal{V}$ be a vector bundle on X. A Harder-Narasimhan filtration is a filtration of subbundles

$$0 = \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \mathcal{V}_2 \subsetneq ... \subsetneq \mathcal{V}_r = \mathcal{V}$$

such that $\mathcal{V}_i/\mathcal{V}_{i-1}$ is semistable with slope λ_i and

$$\lambda_1 > ... > \lambda_r$$

Theorem 4.1.11. *Each vector bundle on X posses a Harder-Narasimhan filtration.*

Proof. [HL10, Theorem 1.3.4] □

The following Lemmas are a special case of a more general statement in [FF18]. We can apply these results, since

$$\pi : \mathbb{P}^1_{\mathbb{C}} \to X$$

is a finite étale Galois covering.

Lemma 4.1.12. *A vector bundle $\mathcal{V}$ on X is semistable if and only if $\pi^*\mathcal{V}$ on $\mathbb{P}^1_{\mathbb{C}}$ is semistable.*

Proof. [FF18, Lemme 5.6.19.]. □

Lemma 4.1.13. *A vector bundle $\mathcal{V}$ on X is pure if and only if $\pi^*\mathcal{V}$ on $\mathbb{P}^1_{\mathbb{C}}$ is pure.*

Proof. The argument given in the proof of [FF18, Proposition 5.6.25.] applies in our case with the slight modification that we have to use Lemma 4.1.7. instead of [FF18, Proposition 5.6.23.]. □

Corollary 4.1.14. *A vector bundle on X is pure if and only if it is semistable.*

Proof. A vector bundle on $\mathbb{P}^1_{\mathbb{C}}$ is semistable if and only if it is pure. By the above two Lemmas this implies the required statement. □

Theorem 4.1.15. *The Harder-Narasimhan filtration is split, i.e. each vector bundle on X is isomorphic to a direct sum $\bigoplus_{i=1}^{n} \mathcal{O}_X(\frac{\lambda_i}{2})^{\oplus d_i}$.*

Proof. Let $\mathcal{V}$ be a vector bundle on X. It comes with a Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \mathcal{V}_2 \subsetneq ... \subsetneq \mathcal{V}_r = \mathcal{V}.$$

The proof will proceed by induction on the index of the filtration. We will show that

$$\mathcal{V}_i \cong \bigoplus_{n > \lambda_{i+1}} \mathcal{O}_X(n)^{\oplus a_n}.$$

- The base case $i = 1$ is covered by Corollary 4.1.14, since $\mathcal{V}_1$ is semistable of slope λ_1 and thus isomorphic to $\mathcal{O}_X(\lambda_1)^{\oplus a_{\lambda_1}}$
- Now for the induction step we use the short exact sequence

$$0 \to \mathcal{V}_i \to \mathcal{V}_{i+1} \to \mathcal{V}_{i+1}/\mathcal{V}_i \to 0.$$

Again by Corollary 4.1.14 and by the induction hypothesis this sequence can be written as

$$0 \to \bigoplus_{n > \lambda_{i+1}} \mathcal{O}_X(n)^{\oplus a_n} \to \mathcal{V}_{i+1} \to \mathcal{O}_X(\lambda_{i+1})^{\oplus a_{\lambda_{i+1}}} \to 0.$$

By Serre duality we have

$$\text{Ext}^1((\mathcal{V}_{i+1}/\mathcal{V}_i), \mathcal{V}_i) \cong H^1(X, \mathcal{V}_i \otimes (\mathcal{V}_{i+1}/\mathcal{V}_i)^\vee) \cong H^0(X, \mathcal{V}_i^\vee \otimes (\mathcal{V}_{i+1}/\mathcal{V}_i) \otimes \mathcal{O}_X(-1)) = 0.$$

The last equality follows, since all the direct summands in $\mathcal{V}_i^\vee \otimes (\mathcal{V}_{i+1}/\mathcal{V}_i) \otimes \mathcal{O}_X(-1)$ are negative and these have trivial global sections. Thus the sequence is split and

$$\mathcal{V}_{i+1} \cong \mathcal{V}_i \bigoplus \mathcal{V}_{i+1}/\mathcal{V}_i \cong \bigoplus_{n > \lambda_{i+1}} \mathcal{O}_X(n)^{\oplus a_n} \bigoplus \mathcal{O}_X(\lambda_{i+1})^{\oplus a_{\lambda_{i+1}}} \cong \bigoplus_{n > \lambda_{i+2}} \mathcal{O}_X(n)^{\oplus a_n}. \qquad \square$$

4.2 Vector bundles via $\text{Isoc}_{\mathbb{R}}$

The classification of vector bundles on X will come down to classifying descent data of vector bundles of $\mathbb{P}^1_{\mathbb{C}}$. Since in our case the morphism $\pi : \mathbb{P}^1_{\mathbb{C}} \to X$ is a Galois covering (with Galois group $\mathbb{Z}/2\mathbb{Z}$), every descent datum of a vector bundle is effective. We get an equivalence of categories between Bun_X, the category of vector bundles on X, and $\text{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X)$, the category of descent data of vector bundles.

Definition 4.2.1. The category of descent data $\text{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X)$ has

- As objects pairs $(\mathcal{V}, v)$, where $\mathcal{V}$ is a vector bundle and v an isomorphism

$$v : f^*\mathcal{V} \to \mathcal{V} \quad \text{such that} \quad v \circ f^*v = \text{id},$$

- As morphisms vector bundle morphisms that commute with the respective descent data.

Let us connect this definition with the classical notion of a descent datum. We have two morphisms

$$p_1, p_2 : \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \times \mathbb{P}^1_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} \coprod \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}} \quad \text{where} \quad p_1 = \mathrm{id} \coprod \mathrm{id} \quad \text{and} \quad p_2 = \mathrm{id} \coprod f.$$

For the classic notion of a descent datum we require a vector bundle $\mathcal{V}$ with an isomorphism

$$p_1^* \mathcal{V} \xrightarrow{\sim} p_2^* \mathcal{V}$$

fulfilling a cocycle condition. By comparing the isomorphism on connected components, this is equivalent to an isomorphism $v : f^* \mathcal{V} \to \mathcal{V}$. In a similar way one sees that the cocycle condition can be expressed as

$$v \circ f^* v = \mathrm{id}\,.$$

Theorem 4.2.2. *The descent data are effective, i.e. the functor is an*

$$\mathrm{Bun}_X \to \mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X)$$

equivalence of categories.

Proof. Combine [Sta18, Tag 0CDQ] with [Gro65, Proposition 2.5.2] □

Example 4.2.3. We denote by D_0 the divisor (0) and by D_∞ the divisor (∞) in $\mathbb{P}^1_{\mathbb{C}}$. We have an isomorphism of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$ modules $\lambda z : \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(D_0) \to \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(D_\infty)$ given by multiplication with $\lambda z \in K(\mathbb{P}^1_{\mathbb{C}})$, where $\lambda \in \mathbb{C}^*$. We now calculate the pullback of this morphism by f^*. By abuse of notation we will denote by $\mathbf{f}$ the induced ring morphism on the graded rings or their localisations.

First let's see that there is a canonical isomorphism $\delta : f^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(D_0) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(D_\infty)$. On $W_1 = \mathrm{Spec}(\mathbb{C}[z])$ and $W_2 = \mathrm{Spec}(\mathbb{C}[\frac{1}{z}])$ we have:

$$f^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(D_0)(W_2) = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(D_0)(W_1) \otimes_{\mathbb{C}[z]} \mathbb{C}[\tfrac{1}{z}] = (z)^{-1}\mathbb{C}[z] \otimes_{\mathbb{C}[z]} \mathbb{C}[\tfrac{1}{z}] \cong (\tfrac{1}{z})^{-1}\mathbb{C}[\tfrac{1}{z}],$$

where the last isomorphism is given by $\mathbf{f} \otimes \mathrm{id}$. Similarly one sees that $f^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(D_0)(W_1) \cong \mathbb{C}[z]$ via $\mathbf{f} \otimes \mathrm{id}$ as well; these isomorphisms agree on $W_0 \cap W_1$ and give a global isomorphism $\delta : f^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(D_0) \to \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(D_\infty)$. More generally the above can be generalized to give isomorphism

$$\delta_n : f^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(nD_0) \to \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(nD_\infty).$$

Now for the pullback on W_1 we have:

$$(z)^{-1}\mathbb{C}[z] \otimes_{\mathbb{C}[z]} \mathbb{C}[\tfrac{1}{z}] \to \mathbb{C}[z] \otimes_{\mathbb{C}[z]} \mathbb{C}[\tfrac{1}{z}], \qquad a \otimes h \mapsto a \cdot (\lambda z) \otimes h = a \otimes -\bar{\lambda}\frac{1}{z} h$$

and repeating the calculation for W_2 shows that $f^*(\lambda z) = -\frac{\bar{\lambda}}{z} : \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(D_\infty) \to \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(D_0)$.

Lemma 4.2.4. *Let $\mathcal{V}$ and $\mathcal{W}$ be semistable vector bundles on $P^1_{\mathbb{C}}$ of the same slope. Then for a closed point $x \in \mathbb{P}^1_{\mathbb{C}}$*

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}}(\mathcal{V}, \mathcal{W}) \cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_x / m_x \mathcal{V}_x, \mathcal{W}_x / m_x \mathcal{W}_x).$$

Proof. We can reduce to the case of semistable vector bundles of slope 0. In this case one can conclude by using that $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) = \mathbb{C}$ and that the category of vector bundles is additive. □

Lemma 4.2.5. *Let (V_n, ϕ_n) be a pure isocrystal of degree n. We write*

$$\mathcal{V}_n = V_n \otimes \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(nD_0), \qquad \mathcal{V}_n^\infty = V_n \otimes \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(nD_\infty).$$

Define the isomorphism $\varphi_n : f^*\mathcal{V}_n \to \mathcal{V}_n$ via the following diagram:

$$\begin{array}{ccc} & \mathcal{V}_n^\infty & \\ {\scriptstyle \phi_n\otimes\delta_n}\nearrow & & \searrow{\scriptstyle \mathrm{id}\otimes\cdot\frac{1}{z^n}} \\ f^*(\mathcal{V}_n) & \xrightarrow{\varphi_n} & \mathcal{V}_n. \end{array}$$

Then $(\mathcal{V}_n, \varphi_n)$ is a descent datum, i.e. an object in $\mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X)$.

Proof. First of all note that $\varphi_n : f^*(\mathcal{V}_n) \to \mathcal{V}_n$ is a morphism of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$ modules, as δ_n is. The main part of the Lemma is to verify that $\varphi_n \circ f^*\varphi_n = \mathrm{id}$. We consider the composition:

$$\begin{array}{ccccccc} & f^*(\mathcal{V}_n^\infty) & & & \mathcal{V}_n^\infty & & \\ {\scriptstyle f^*(\phi_n\otimes\delta_n)}\nearrow & & \searrow{\scriptstyle f^*(\mathrm{id}\otimes\cdot\frac{1}{z^n})} & {\scriptstyle \phi_n\otimes\delta_n}\nearrow & & \searrow{\scriptstyle \mathrm{id}\otimes\cdot\frac{1}{z^n}} & \\ f^*(f^*(\mathcal{V}_n)) & & \xrightarrow{f^*\varphi_n} & f^*(\mathcal{V}_n) & & \xrightarrow{\varphi_n} & \mathcal{V}_n \end{array}$$

We evaluate the commutative triangles at W_1:

$$\begin{array}{ccc} & & V_n \otimes z^n\mathbb{C}[\frac{1}{z}] \otimes_{\mathbb{C}[\frac{1}{z}]} \mathbb{C}[z] \\ & {\scriptstyle f^*(\phi\otimes\delta_n)}\nearrow & \downarrow{\scriptstyle \mathrm{id}\otimes\cdot\frac{1}{z^n}\otimes\mathrm{id}} \\ V_n \otimes \frac{1}{z^n}\mathbb{C}[z] & \xrightarrow{f^*\varphi_n} & V_n \otimes \mathbb{C}[\frac{1}{z}] \otimes_{\mathbb{C}[\frac{1}{z}]} \mathbb{C}[z] \end{array} \qquad \begin{array}{ccc} & & V_n \otimes \mathbb{C}[\frac{1}{z}] \\ & {\scriptstyle \phi_n\otimes\delta_n}\nearrow & \downarrow{\scriptstyle \mathrm{id}\otimes\cdot\frac{1}{z^n}} \\ V_n \otimes \mathbb{C}[\frac{1}{z}] \otimes_{\mathbb{C}[\frac{1}{z}]} \mathbb{C}[z] & \xrightarrow{\varphi_n} & V_n \otimes \frac{1}{z^n}\mathbb{C}[z] \end{array}$$

Here we identified $f^*(f^*(\mathcal{V}_n))$ canonically with $(f \circ f)^*(\mathcal{V}_n) = \mathcal{V}_n$. Thus we see that on W_1 we have

$$f^*\varphi_n(v \otimes g(z)) = \phi_n(v) \otimes \bar{g}\left(-\frac{1}{z}\right) \cdot \frac{1}{z^n} \otimes 1$$

and furthermore evaluating at the right triangle gives

$$\varphi_n(\phi_n(v) \otimes \bar{g}\left(-\frac{1}{z}\right) \cdot \frac{1}{z^n} \otimes 1) = \phi_n^2(v) \otimes g(z) \cdot (-1)^n = (-1)^{2n} v \otimes g(z) = v \otimes g(z).$$

Now one might repeat the calculation for W_2, which is similar to the above one, or use Lemma 4.2.4. to conclude that indeed $\varphi_n \circ f^*\varphi_n = \mathrm{id}$. □

Construction 4.2.6. With the above Lemma we can construct a functor

$$\mathcal{E} : \mathrm{Isoc}_{\mathbb{R}} \to \mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X).$$

To each isocrystal $(\bigoplus_{n\in\mathbb{Z}} V_n, \bigoplus_{n\in\mathbb{Z}} \phi_n)$ we associate the vector bundle

$$(\bigoplus_{n\in\mathbb{Z}} \mathcal{V}_n, \bigoplus_{n\in\mathbb{Z}} \varphi_n).$$

This can be made functorial in the obvious way.

Theorem 4.2.7. *The essential image of the functor* $\mathcal{E} : \mathrm{Isoc}_{\mathbb{R}} \to \mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X)$ *contains the pure objects, i.e.* $(\mathcal{V}, v)$ *such that* $\mathcal{V} \cong \mathcal{O}(nD)^{\oplus r}$*.*

Proof. As stated above $\mathcal{V}$ admits an isomorphism $\psi : \mathcal{V} \to \mathcal{O}(nD_0)^{\oplus r}$ which induces an isomorphism of descent data $(\mathcal{V}, v) \cong (\mathcal{O}(nD_0)^{\oplus r}, \psi \circ v \circ \psi^{-1})$. So without loss of generality we assume that $\mathcal{V} = V \otimes \mathcal{O}(nD_0)$ with $\dim(V) = r$.

The morphism $v : f^*\mathcal{V} \to \mathcal{V}$ induces a morphism on stalks at the point $p = [1 : 1]$. We can identify

$$(f^*\mathcal{V})_p \cong \mathcal{V}_{f(p)} \otimes_{\mathcal{O}_{X,f(p)}} \mathcal{O}_{X,p} \cong (V \otimes \mathcal{O}_{X,f(p)}) \otimes_{\mathcal{O}_{X,f(p)}} \mathcal{O}_{X,p}$$

and after we modulo out by m_p we are left with a morphism of $\mathbb{C}$ vector spaces $v_p : V \otimes_\sigma \mathbb{C} \to V$. Here the subscript at the tensor product means that we tensor via the complex conjugation. Finally we can define the antiholomorphic automorphism as the following composition

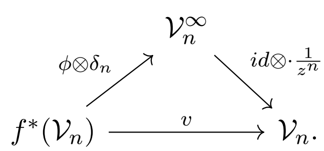
$$\begin{aligned} \phi \colon V &\to V \otimes_\sigma \mathbb{C} \to V \\ w &\mapsto w \otimes_\sigma 1 \mapsto v_p(w \otimes 1). \end{aligned}$$

Let's verify that this is indeed an isocrystal. First note that for $\lambda \in \mathbb{C}$

$$\phi(\lambda w) = v_p(\lambda w \otimes_\sigma 1) = v_p(w \otimes_\sigma \bar{\lambda}) = \bar{\lambda} v_p(w \otimes_\sigma 1) = \bar{\lambda}\phi(w)$$

where in the third equality we use that v_p is morphism of $\mathcal{O}_{X,p}/m_p$-modules.

Having established that ϕ is antiholomorphic, the morphism $\phi \otimes \delta_n : f^*(\mathcal{V}_n) \to \mathcal{V}_n^\infty$ is well-defined and we can consider



By construction this diagram commutes at the stalks of the point p. Now by Lemma 4.2.4. this is already sufficient to conclude that the diagram is commutative on $\mathbb{P}^1_{\mathbb{C}}$. A calculation similar to the one in Lemma 4.2.5. allows us to conclude:

$$w \otimes 1 = v(f^*v(w \otimes 1)) = \phi^2(w) \otimes (-1)^n$$

where now the first equality is a consequence of $v : f^*V \to V$ being a descent datum. Thus we have proven that (V, ϕ) is an isocrystal and the commutativity of the above triangle shows that $\mathcal{E}(V, \phi) = (\mathcal{V}, v)$. ☐

Corollary 4.2.8. *The functor* $\mathcal{E} : \mathrm{Isoc}_{\mathbb{R}} \to \mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X)$ *is essentially surjective.*

Proof. By Theorem 4.2.7. the essential image contains pure vector bundles, hence they contain also direct sums of vector bundles. By Theorem 4.1.15 every vector bundle is isomorphic to a direct sum of pure vector bundles. Thus $\mathcal{E}$ is essentially surjective. ☐

Corollary 4.2.9. *The functor* $\mathcal{E} : \mathrm{Isoc}_{\mathbb{R}} \to \mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X)$ *is faithful. When restricted to the subcategory of pure isocrystals of slope* n *for some* $n \in \mathbb{Z}$*, it is also full.*

Proof. The functor is faithful as we can compare both morphisms at the stalk $1 \in \mathbb{P}^1_{\mathbb{C}}$ and recover the original morphism in $\mathrm{Isoc}_{\mathbb{R}}$.

Now assume we have two pure isocrystals (V, ϕ_V) and (W, ϕ_W) of the same slope. In this case applying $\mathcal{E}$ will yield two pure, or equivalently semistable, vector bundles. Thus we can apply Lemma 4.2.4., which implies that a morphism $\mathcal{E}(V, \phi_V) \to \mathcal{E}(W, \phi_W)$ is given by a linear morphism $f \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ that commutes with descent data, which is exactly the condition

$$f \circ \phi_V = \phi_W \circ f.$$

□

Chapter 5

$B(G,\mathbb{R})$ as G-bundles on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$

The aim of this chapter is to explore the relationship between $B(G,\mathbb{R})$ and G-bundles on $X := \widetilde{\mathbb{P}^1_{\mathbb{R}}}$. In particular we want to prove

Theorem 5.0.1. *There exists an isomorphism $H^1_{\acute{e}t}(X,G) \cong B(G,\mathbb{R})$.*

This theorem can be seen as the descent version of the isomorphism $H^1_{\acute{e}t}(\mathbb{P}^1_{\mathbb{C}},G) \cong B(G,\mathbb{C})$ in the complex case; there is a similiar analogy for the Fargues-Fontaine curve in the non-archimedean case.

The proof proceeds in two steps:

$$B(G,\mathbb{R}) \xleftrightarrow{(1)} \mathrm{Hom}^{\otimes}(\mathrm{Rep}_{\mathbb{R}}(G),\mathrm{Isoc}_{\mathbb{R}}) \xleftrightarrow{(2)} H^1_{\acute{e}t}(X,G)$$

The first equivalence was proven in Section 3.4. After proving the second equivalence, we will show that under this identification we have

$$B(G,\mathbb{R})_{bsc} \cong \{\text{semistable } G\text{-bundles}\}/\sim .$$

5.1 $\mathrm{Hom}^{\otimes}(\mathrm{Rep}_{\mathbb{R}}(G),\mathrm{Isoc}_{\mathbb{R}})$ and $H^1_{\acute{e}t}(X,G)$

The main reference for this section is [Ans18]. We start by introducing some notation from [Zie11].

Definition 5.1.1. A Tannakian category over k is an essentially small symmetric monoidal category T which is abelian, k-linear and rigid, for which the natural morphism $k \to \mathrm{End}(1)$ induced by the k-linear structure of T is an isomorphism and for which there exists a non-empty scheme S over k and an exact k-linear tensor functor ω from T to $\mathrm{QCoh}(S)$.

Example 5.1.2. The main example for us in this section will be $\mathrm{Isoc}_{\mathbb{R}}$. It is easy to see that we can endow it with a tensor product, which fulfills all the above requirements. Furthermore we can embed $\mathrm{Isoc}_{\mathbb{R}}$ into $\mathrm{Qcoh}(X)$ by the results of the last chapter.

Theorem 5.1.3. *The functor $\mathcal{E} : \mathrm{Isoc}_{\mathbb{R}} \to \mathrm{Bun}_X$ is an exact faithful tensor functor inducing a bijection on isomorphism classes.*

Proof. By Corollary 4.2.8 the functor $\mathcal{E}$ is essentially surjective and by Corollary 4.2.9. it is faithful. Thus it is a bijection on isomorphism classes.

Now we show that $\mathcal{E}$ is a tensor functor. Since $\mathcal{E}$ preserves direct sums, it suffices to show that for two pure isocrystals (V_n, ϕ_n) and (W_m, α_m) the tensor product is preserved. As the construction

$$\mathrm{Isoc}_{\mathbb{R}} \to \mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to \widetilde{\mathbb{P}^1_{\mathbb{R}}})$$

preserves tensor products, it remains to see that under the equivalence

$$\mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to \widetilde{\mathbb{P}^1_{\mathbb{R}}}) \cong \mathrm{Bun}_X$$

the tensor product is preserved. To prove this, it suffices to show that for all vector bundles $\mathcal{V}, \mathcal{W}$ on X, there is a natural isomorphism

$$\alpha_{\mathcal{V},\mathcal{W}} : \pi^*(\mathcal{W} \otimes_{\mathcal{O}_X} \mathcal{V}) \cong \pi^*\mathcal{W} \otimes_{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}} \pi^*\mathcal{V}$$

which respects the descent data. We may work locally to construct such an isomorphism. In this case the claim follows from the following argument. Let A be an $\mathbb{R}$-algebra with A-modules M and N. Then the chain of isomorphisms

$$(M \otimes_A N) \otimes_{\mathbb{R}} \mathbb{C} \cong M \otimes_A N_{\mathbb{C}} \cong M_{\mathbb{C}} \otimes_{A_{\mathbb{C}}} N_{\mathbb{C}}$$

respects the descent data, i.e. the antiholomorphic involutions.

Finally let us show exactness. Since $\mathrm{Hom}_{\mathrm{Isoc}_{\mathbb{R}}}(V, W) = 0$ for isocrystals of different slopes, we are reduced to show that exact sequences consisting of pure isocrystals are preserved. In this case it follows from Corollary 4.2.9., since $\mathcal{E}\mid_{\mathrm{Isoc}^n_{\mathbb{R}}}$ is an equivalence. □

Remark 5.1.4. As in the case with $\mathbb{P}^1_{\mathbb{C}}$, the functor is not an equivalence nor exact. For example the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2) \to \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \to 0$$

descends to

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X(-\frac{1}{2}) \to \mathcal{O}_X \to 0.$$

Definition 5.1.5. Let X be a scheme. Denote by FilBun_X is the category of locally free sheaves $\mathcal{V}$ with a decreasing filtration by subbundles $F^*\mathcal{V}$ satisfying:

1. $\bigcup_{n\in\mathbb{Z}} F^n\mathcal{V} = \mathcal{V}$,
2. $\bigcap_{n\in\mathbb{Z}} F^n\mathcal{V} = 0$.

The morphisms are given by $f : \mathcal{V} \to \mathcal{W}$ such that $f(F^n\mathcal{V}) \subset F^n\mathcal{W}$.

We would like to endow this category with a certain notion of exactness, for more details [Zie11, Chapter 4].

Definition 5.1.6. A morphism $f : \mathcal{V} \to \mathcal{W}$ in FilBun_X is called admissible, if $f(F^n\mathcal{V}) = Im(f) \cap F^n\mathcal{W}$.
A sequence in FilBun_X is called short exact, if the underlying sequence of sheaves is short exact and the morphisms are admissible.

Construction 5.1.7. Recall from the last chapter that given a vector bundle on $\mathcal{V}$ on X, we can find the Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \mathcal{V}_2 \subsetneq ... \subsetneq \mathcal{V}_r = \mathcal{V}.$$

For indexing reasons we rather want a filtration indexed by $\frac{1}{2}\mathbb{Z}$

$$.... \subset \mathrm{HN}^{i+\frac{1}{2}}(\mathcal{V}) \subset \mathrm{HN}^{i}(\mathcal{V}) \subset \mathrm{HN}^{i-\frac{1}{2}}(\mathcal{V}) \subset ...$$

such that

$$\mathrm{HN}^{i}(\mathcal{V}) / \mathrm{HN}^{i+\frac{1}{2}}(\mathcal{V})$$

is semistable of slope i. These notions are equivalent, since it is just a matter of reindexing. For example given a filtration as above, we can set

$$\mathrm{HN}^{i}(\mathcal{V}) = \mathcal{V}_d, \qquad d = min\{k \mid i < \mu_{k+1}\}.$$

Remark 5.1.8. As in [Ans18, p.2] we can easily describe the Harder-Narasimhan filtration of a vector bundle $\mathcal{V} = \mathcal{E}(V, \phi)$. For this decompose the isocrystal into its isocline factors $(V, \phi) = \bigoplus_{i \in \mathbb{Z}} (V_i, \phi_i)$. Then define $\mathrm{fil}^n(V, \phi) = \bigoplus_{i \geq n} (V_i, \phi_i)$ and $\mathrm{HN}^n \mathcal{V} = \mathcal{E}(\mathrm{fil}^n(V, \phi))$.

Lemma 5.1.9. *There is a functor* $\mathrm{HN} : \mathrm{Bun}_X \to \mathrm{FilBun}_X$ *sending a vector bundle to its Harder-Narasimhan filtration. This functor is fully faithful.*

Proof. We already described the construction. Every vector bundle morphism preserves the Harder-Narasimhan filtration, thus we can make the above construction functorial. It is fully faithful, since the Harder-Narasimhan filtration stabilizes eventually, i.e.

$$\mathrm{HN}^{i}(\mathcal{V}) = \mathcal{V}$$

for sufficient big $i \in \frac{1}{2}\mathbb{Z}$. ☐

Definition 5.1.10. The category GrBun_X is the category of vector bundles with a decomposition $\mathcal{V} = \oplus_{n \in \mathbb{Z}} \mathcal{V}_n$ of subbundles. Morphisms are degree preserving morphisms of vector bundles.

Lemma 5.1.11. *The functor* $\mathrm{gr} : \mathrm{FilBun}_X \to \mathrm{GrBun}_X$ *given by*

$$\mathrm{gr}(\mathcal{V}, F^*\mathcal{V}) := \bigoplus_{n \in \mathbb{Z}} F^n\mathcal{V}/F^{n+1}\mathcal{V}$$

is a tensor exact functor.

Proof. For exactness note that if

$$0 \to (\mathcal{V}, F^*\mathcal{V}) \to (\mathcal{W}, F^*\mathcal{W}) \to (\mathcal{Z}, F^*\mathcal{Z}) \to 0$$

is exact in FilBun_X, then

$$0 \to F^p\mathcal{V} \to F^p\mathcal{W} \to F^p\mathcal{Z} \to 0$$

is exact again. Then we can conclude by 3×3 Lemma. The admissibility condition is crucial here. The tensor part follows from

$$\sum_{p+q=n} F^p\mathcal{V} \otimes F^q\mathcal{W} \Big/ \sum_{p+q=n+1} F^p\mathcal{V} \otimes F^q\mathcal{W} \cong \bigoplus_{p+q=n} F^p\mathcal{V}/F^{p+1}\mathcal{W} \otimes F^q\mathcal{V}/F^{q+1}\mathcal{W}$$

☐

Lemma 5.1.12. *The composite functor*

$$\mathrm{Isoc}_{\mathbb{R}} \xrightarrow{\mathcal{E}} \mathrm{Bun}_X \xrightarrow{\mathrm{HN}} \mathrm{FilBun}_X \xrightarrow{gr} \mathrm{GrBun}_X$$

is an equivalence of exact categories from $\mathrm{Isoc}_{\mathbb{R}}$ *to its essential image in* GrBun_X*, which consists of graded vector bundles*

$$\mathcal{E} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathcal{E}_i$$

such that $\mathcal{E}_i$ *is semistable of slope* i*.*

Proof. Essential surjectivity is straightforward since semistable vector bundles are exactly the pure vector bundles.

The functor is fully faithful by Corollary 4.2.9. □

Remark 5.1.13. We will denote the inverse of the composite functor by $\mathcal{E}_{gr}^{-1}$.

Definition 5.1.14. Let $\mathcal{G}$ be a sheaf of groups on $(Sch/X)_{ét}$. A $\mathcal{G}$-torsor $\mathcal{P}$ over X (for the étale topology) is a sheaf on $(Sch/X)_{ét}$ with a left $\mathcal{G}$-action $\mathcal{G} \times \mathcal{P} \to \mathcal{P}$ such that there exists an étale cover $\{U_i \to X\}$ on which there exist G-equivariant isomorphisms $\mathcal{P} \mid_{U_i} \cong \mathcal{G} \mid_{U_i}$.

Lemma 5.1.15. *Let* G *be a reductive group over* $\mathbb{R}$*. Sending a* G*-torsor* $\mathcal{P}$ *over* X *to*

$$\omega : \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Bun}_X, \qquad V \mapsto \mathcal{P} \times^G (V \otimes_{\mathbb{R}} \mathcal{O}_X)$$

defines an equivalence from the groupoid of G*-torsors to the groupoid of exact tensor functors from* $\mathrm{Rep}_{\mathbb{R}}(G)$ *to* Bun_X*. The inverse equivalence sends an exact tensor functor* $\omega : \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Bun}_X$ *to the* G*-torsor* $\mathrm{Isom}(\omega_{can}, \omega)$ *of isomorphisms of* ω *to the canonical fiber functor*

$$\omega_{can} : \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Bun}_X, \qquad V \mapsto V \otimes_{\mathbb{R}} \mathcal{O}_X.$$

Proof. [Far, Section 4.1] □

Remark 5.1.16. We will refer to G-torsors also as G-bundles.

Definition 5.1.17. Let

$$\omega : \mathcal{T} \to \mathrm{FilBun}_X$$

be an exact tensor functor from a Tannakian category $\mathcal{T}$.

- A splitting of ω is a functor

  $$\gamma : \mathcal{T} \to \mathrm{GrBun}_X$$

  such that $\mathrm{fil} \circ \gamma = \omega$.

- Define the following presheaves on Sch/X:

  1. $\mathrm{Spl}(\omega)(Y \xrightarrow{f} X) := \{$*set of splittings of* $\mathcal{T} \xrightarrow{\omega} \mathrm{Bun}_X \xrightarrow{f^*} \mathrm{Bun}_Y\}$
  2. $U(\omega) := \mathrm{Ker}(\mathrm{Aut}^{\otimes}(\omega) \to \mathrm{Aut}^{\otimes}(gr \circ \omega))$

Lemma 5.1.18.
- *With the above notation $U(\omega)$ is representable by an affine group scheme over k. It comes together with a decreasing filtration of normal subgroups*

$$U(\omega) = U_1(\omega) \supseteq U_2(\omega) \supseteq ...U_i(\omega) \supseteq ...$$

for $i \geq 1$. Furthermore the quotient $gr^i U(\omega) := U_i(\omega)/U_{i+1}(\omega)$ is abelian and isomorphic to

$$\operatorname{gr}^i U(\omega) \cong \operatorname{Lie}(\operatorname{gr}^i U(\omega)) \cong \operatorname{gr}^i(\omega(\operatorname{Lie}(G))).$$

- *Spl(ω) is representable by an $U(\omega)$-torsor for the fpqc-topology.*

Proof. [Zie11, Section 4.3] □

Lemma 5.1.19. *Let*

$$\omega : \operatorname{Rep}_{\mathbb{R}}(G) \to \operatorname{Bun}_X$$

be an exact tensor functor. Then

$$\operatorname{HN} \circ \omega : \operatorname{Rep}_{\mathbb{R}}(G) \to \operatorname{FilBun}_X$$

is still exact.

Proof. As we only consider reductive groups over characteristic zero fields, we can apply [DM18]. This tells us that $\operatorname{Rep}_{\mathbb{R}}(G)$ is semisimple and in particular every short exact sequence

$$0 \to V \to W \to Z \to 0$$

is split. But then

$$\operatorname{HN}(\omega(V)) \to \operatorname{HN}(\omega(W)) \to \operatorname{HN}(\omega(Z))$$

is also split. Thus $\operatorname{HN} \circ \omega$ is exact. □

Definition 5.1.20. Let $\underline{\operatorname{Hom}}^{\otimes}(\operatorname{Rep}_{\mathbb{R}}(G), \operatorname{Isoc}_{\mathbb{R}})$ be the groupoid of exact tensor functors

$$\operatorname{Rep}_{\mathbb{R}}(G) \to \operatorname{Isoc}_{\mathbb{R}}$$

and $\operatorname{Hom}^{\otimes}(\operatorname{Rep}_{\mathbb{R}}(G), \operatorname{Isoc}_{\mathbb{R}})$ be the set of isomorphism classes.
Similarly define $\underline{\operatorname{Hom}}^{\otimes}(\operatorname{Rep}_{\mathbb{R}}(G), \operatorname{Bun}_X)$ and $\operatorname{Hom}^{\otimes}(\operatorname{Rep}_{\mathbb{R}}(G), \operatorname{Bun}_X)$.

Theorem 5.1.21. *Let G be a reductive group over $\mathbb{R}$. Then composition with $\mathcal{E}$ defines a faithful functor*

$$\Phi : \underline{\operatorname{Hom}}^{\otimes}(\operatorname{Rep}_{\mathbb{R}}(G), \operatorname{Isoc}_{\mathbb{R}}) \to \underline{\operatorname{Hom}}^{\otimes}(\operatorname{Rep}_{\mathbb{R}}(G), \operatorname{Bun}_X)$$

which induces a bijection

$$\operatorname{Hom}^{\otimes}(\operatorname{Rep}_{\mathbb{R}}(G), \operatorname{Isoc}_{\mathbb{R}}) \to H^1_{\acute{e}t}(X, G)$$

on isomorphism classes.

Proof. By Lemma 5.1.12.

$$\mathrm{Isoc}_{\mathbb{R}} \xrightarrow{\mathcal{E}} \mathrm{Bun}_X \xrightarrow{\mathrm{HN}} \mathrm{FilBun}_X \xrightarrow{gr} \mathrm{GrBun}_X$$

is an equivalence onto its essential image. In particular the functor

$$\Phi : \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Isoc}_{\mathbb{R}}) \to \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Bun}_X)$$

is faithful and induces an injection on isomorphism classes. Thus the only thing left to prove that each exact tensor functor

$$\omega : \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Bun}_X$$

factors as $\omega \cong \mathcal{E} \circ \omega'$, where

$$\omega' : \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Isoc}_{\mathbb{R}}$$

is an exact tensor functor.
Consider

$$\tilde{\omega} : \mathrm{Rep}_{\mathbb{R}}(G) \xrightarrow{\omega} \mathrm{Bun}_X \xrightarrow{\mathrm{HN}} \mathrm{FilBun}_X .$$

This is an exact tensor functor by Lemma 5.1.19. and thus we can apply Lemma 5.1.18. In particular we get a series

$$U(\tilde{\omega}) = U_1(\tilde{\omega}) \supseteq U_2(\tilde{\omega}) \supseteq ...$$

such that

$$\mathrm{gr}^i U(\tilde{\omega}) \cong \mathrm{gr}^i \tilde{\omega}(\mathrm{Lie}(G))$$

are semistable vector bundles of rank $i > 0$, i.e. $\mathrm{gr}^i U(\tilde{\omega}) \cong \mathcal{O}_X(i)^d$. By Lemma 4.1.7. we have $H^1_{ét}(X, \mathrm{gr}^i U(\tilde{\omega})) = 0$. With the short exact sequence

$$0 \longrightarrow U_{i+1}(\tilde{\omega}) \longrightarrow U_i(\tilde{\omega}) \longrightarrow \mathrm{gr}^i(U(\tilde{\omega})) \longrightarrow 0$$

we can argue inductively to conclude that $H^1_{ét}(X, U(\tilde{\omega})) = 0$. But this means that the the $U(\tilde{\omega})$-torsor $\mathrm{Spl}(\tilde{\omega})$ is already trivial, i.e. there exists a splitting

$$\gamma : \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{GrBun}_X$$

over X. Since $\gamma \cong \mathrm{gr} \circ \tilde{\omega}$, we get an induced natural isomorphism

$$\alpha(V, r) : \omega(V, r) \to \bigoplus_{i \in \mathbb{Z}} \mathrm{gr}^i \tilde{\omega}(V, r)$$

between the vector bundle $\omega(V, r)$ and the associated graded vector bundle of its Harder-Narasimhan filtration. Now objects of the form $\bigoplus_{i \in \mathbb{Z}} \mathrm{gr}^i \tilde{\omega}(V, r)$ can be regarded as isocrystals through the equivalence of $\mathrm{Isoc}_{\mathbb{R}}$ into its essential image (cf. Lemma 5.1.13.). Thus we may define

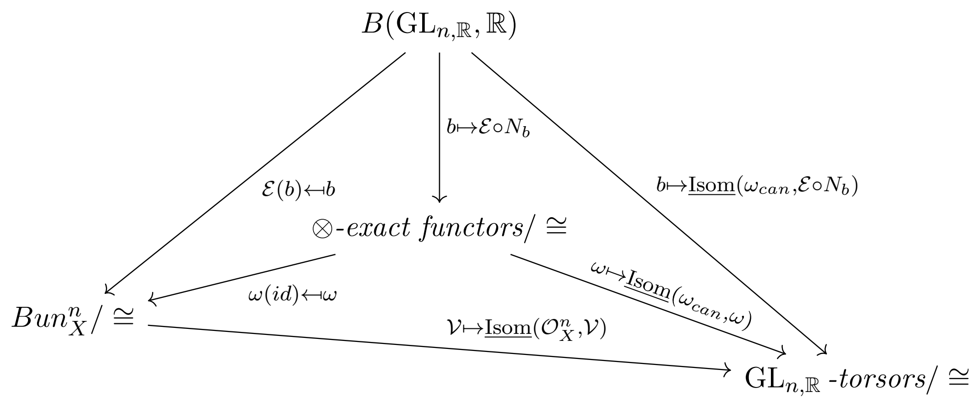
$$\omega' : \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Isoc}_{\mathbb{R}}, \qquad (V, r) \mapsto \mathcal{E}_{gr}^{-1}(\bigoplus_{i \in \mathbb{Z}} \mathrm{gr}^i \tilde{\omega}(V, r)).$$

The natural isomorphism α then induces a natural isomorphism

$$\omega \cong \mathcal{E} \circ \omega'$$

and this is exactly what we wanted. ∎

Lemma 5.1.22. *The following diagram commutes*



(5.1)

Proof. The left triangle commutes by our construction of N_b and the right triangle commutes trivially. It remains to show that the lower triangle commutes. Thus we have to prove that

$$\underline{\mathrm{Isom}}(\omega_{can}, \omega) \xrightarrow{\sim} \underline{\mathrm{Isom}}(\mathcal{O}_X^n, \omega(id)).$$

It suffices to find a morphism of torsors. Such a morphism is given by

$$\begin{aligned} \alpha(U) : \underline{\mathrm{Isom}}(\omega_{can}, \omega)(U) &\xrightarrow{\sim} \underline{\mathrm{Isom}}(\mathcal{O}_X^n, \omega(id))(U) \\ \theta &\mapsto \theta(\mathbb{R}^n, id). \end{aligned}$$

□

5.2 Basic elements and semistable G-bundles

Throughout this section G is a reductive group over $\mathbb{R}$ and we write $\mathfrak{g}$ for $\mathrm{Lie}(G)$.

Definition 5.2.1. Let $\mathcal{P}$ be a G-torsor over X. $\mathcal{P}$ is called semistable, if the adjoint bundle

$$\mathrm{Ad}(\mathcal{P}) = \mathcal{P} \times^{G,\mathrm{Ad}} \mathfrak{g}$$

is semistable.

Proposition 5.2.2. If $\mathrm{Ad}(\mathcal{P})$ is semistable, it has slope 0.

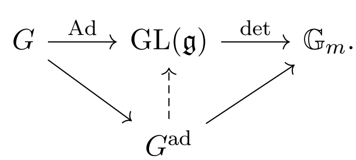
Proof. First note that

$$\mu(\mathrm{Ad}(\mathcal{P})) = \mathrm{rk}(\mathrm{Ad}(\mathcal{P})) \cdot \mu(\det(\mathrm{Ad}(\mathcal{P})))$$

by definition of the degree. Thus it is enough to show that

$$\mu(\det(\mathrm{Ad}(\mathcal{P}))) = \deg(\det(\mathrm{Ad}(\mathcal{P}))) = 0.$$

The adjoint representation factors through its adjoint group G^{ad}, i.e. we have a commutative diagram



Thus $\det \circ \mathrm{Ad}$ is trivial, since every morphism

$$G^{\mathrm{ad}} \to \mathbb{G}_m$$

is. We can conclude since

$$\det(\mathrm{Ad}(\mathcal{P})) \cong \det(\mathcal{P} \times^{\mathrm{Ad}} \mathrm{GL}(\mathfrak{g}) \times^{\mathrm{GL}(\mathfrak{g})} \mathfrak{g}) \cong \mathcal{P} \times^{\mathrm{Ad}} \mathrm{GL}(\mathfrak{g}) \times^{\det} \bigwedge^{\dim(\mathfrak{g})} \mathfrak{g} \cong \mathcal{P} \times^{\det \circ \mathrm{Ad}} \bigwedge^{\dim(\mathfrak{g})} \mathfrak{g} \cong \mathcal{P} \times^{e} \bigwedge^{\dim(\mathfrak{g})} \mathfrak{g}$$

implies that $\det(\mathrm{Ad}(\mathcal{P})) \cong \mathcal{P} \times^e \bigwedge^{\dim(\mathfrak{g})} \mathfrak{g} \cong \mathcal{O}_X$ is trivial. □

For the next part we need the following theorem;

Theorem 5.2.3. *Let G be a reductive group over an arbitrary field k. The inclusion*

$$Z(G) \subset ker(\mathrm{Ad})$$

is an equality. In particular we get a faithful representation

$$G^{\mathrm{ad}} \hookrightarrow \mathrm{GL}(\mathrm{Lie}(G)).$$

Proof. [Gro70, Proposition 4.11.] □

Lemma 5.2.4. *An element $b \in B(G, \mathbb{R})$ is basic if and only if $\mathrm{Ad}(b) \in B(\mathrm{GL}(\mathfrak{g}), \mathbb{R})$ is pure (in which case it will be automatically of slope 0).*

Proof. First assume that $b \in B(G, \mathbb{R})$ is basic. Then

$$v_b : \mathbb{G}_m \to G_{\mathbb{C}}$$

has image inside $Z(G_{\mathbb{C}})$. Since

$$\mathrm{im}(Z(G_{\mathbb{C}}) \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathfrak{g}_{\mathbb{C}})) = e,$$

this implies that

$$v_{\mathrm{Ad}(b)} = \mathrm{Ad}(v_b) = e.$$

Thus $\mathrm{Ad}(b)$ is trivial, i.e. pure of slope 0.

Now assume that $\mathrm{Ad}(b)$ is pure. Thus $v_{\mathrm{Ad}(b)}$ factors through the center

$$\begin{array}{ccc} \mathbb{G}_m & & \\ \downarrow & \searrow & \\ Z(\mathrm{GL}(\mathfrak{g}_{\mathbb{C}})) & \hookrightarrow & \mathrm{GL}(\mathfrak{g}_{\mathbb{C}}). \end{array}$$

But since $\det \circ \mathrm{Ad}$ is trivial, we get a commutative diagram

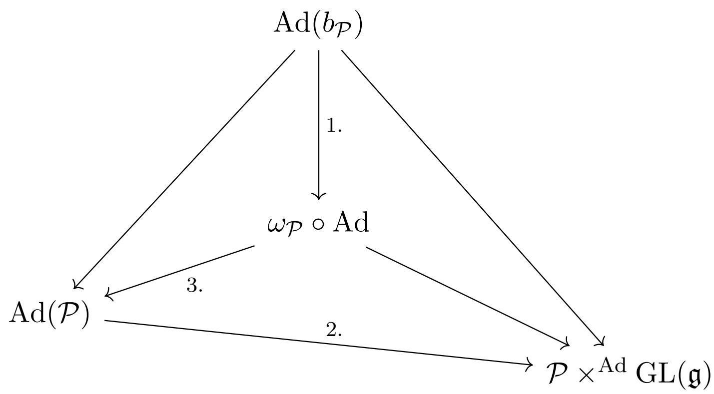
$$\begin{array}{ccccc} & & \mathbb{G}_m & & \\ & \swarrow & \downarrow{\scriptstyle \mathrm{Ad} \circ v_b} & \overset{e}{\searrow} & \\ \mu_{dim(\mathfrak{g}_{\mathbb{C}})} & \hookrightarrow & Z(\mathrm{GL}(\mathfrak{g}_{\mathbb{C}})) & \xrightarrow{\det} & \mathbb{G}_m. \end{array}$$

As there are no non-trivial morphisms $\mathbb{G}_m \to \mu_{\dim(\mathfrak{g}_\mathbb{C})}$, we conclude that $\mathrm{Ad} \circ v_b$ must be trivial. In particular, as the adjoint representation of the adjoint group is faithful by Theorem 5.2.3., the composition

$$\mathbb{G}_{m,\mathbb{R}} \xrightarrow{v_b} G \to G^{\mathrm{ad}}$$

is trivial and by the universal property we see that $\mathbb{G}_m$ factors through $Z(G_\mathbb{C})$. ☐

Lemma 5.2.5. *Let $\mathcal{P}$ be a G-torsor. Then we have the following relations corresponding to the diagram (5.1.)*



Proof. It suffices to prove the three indicated relations.

1. $\mathrm{Ad}(b_\mathcal{P})$ is given by composing $b_\mathcal{P}$ with Ad. Thus the corresponding tensor functor will be given by $\mathcal{E} \circ N_{b_\mathcal{P}} \circ \mathrm{Ad}$ and since $\omega_\mathcal{P} \cong \mathcal{E} \circ N_{b_\mathcal{P}}$, we conclude that $\omega_\mathcal{P} \circ \mathrm{Ad}$ indeed corresponds to $\mathrm{Ad}(b_\mathcal{P})$,
2. This is straightforward from the equivalence between $\mathrm{GL}(\mathfrak{g})$-torsors and vector bundles,
3. We compute
$$(\omega_\mathcal{P} \circ \mathrm{Ad})(\mathfrak{g}, id) = \omega_\mathcal{P}(\mathfrak{g}, \mathrm{Ad}) \cong \mathcal{P} \times^{\mathrm{Ad}} \mathfrak{g} = \mathrm{Ad}(\mathcal{P}).$$

☐

The following proof is an elaborated version of the proof given in [Far17, Proposition 5.12.].

Lemma 5.2.6. *Let $\mathcal{P}$ be a G-torsor over $\widetilde{\mathbb{P}^1_\mathbb{R}}$ and $b_\mathcal{P}$ the corresponding class inside $B(G, \mathbb{R})$. Then $b_\mathcal{P}$ is basic if and only if $\mathcal{P}$ is semistable.*

Proof.
- By definition $\mathcal{P}$ is semistable if and only if $\mathrm{Ad}(\mathcal{P})$ is semistable
- $\mathrm{Ad}(\mathcal{P})$ is semistable if and only if $b_{\mathrm{Ad}(\mathcal{P})}$ is pure; this is true for arbitrary vector bundles
- By Lemma 5.2.5. $b_{\mathrm{Ad}(\mathcal{P})} = \mathrm{Ad}(b_\mathcal{P})$ and thus $b_{\mathrm{Ad}(\mathcal{P})}$ is pure if and only if $\mathrm{Ad}(b_\mathcal{P})$ is pure
- By Lemma 5.2.4. $\mathrm{Ad}(b_\mathcal{P})$ is pure if and only if $b_\mathcal{P}$ is basic.

☐

Theorem 5.2.7. *Let $b \in B(G, \mathbb{R})$. The following are equivalent*

1. *b is basic*
2. *the corresponding torsor $\mathcal{P}$ is semistable*

3. *the morphism corresponding to* $\mathrm{Rep}_{\mathbb{R}}(G) \xrightarrow{N_b} \mathrm{Isoc}_{\mathbb{R}} \xrightarrow{F} \mathrm{Rep}_{\mathbb{R}}(\mathbb{G}_m)$ *is central*

and imply that

4. *there exists a unique splitting* $\gamma : \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{GrBun}_X$ *for the functor* $\tilde{\omega} : \mathrm{Rep}_{\mathbb{R}}(G) \xrightarrow{N_b} \mathrm{Isoc}_{\mathbb{R}} \xrightarrow{\mathcal{E}} \mathrm{Bun}_X \xrightarrow{\mathrm{HN}} \mathrm{FilBun}_X$.

Proof. We showed the equivalence of 1. and 2. above.

To prove the equivalence of 1. and 3. one verifies that the composition

$$\mathrm{Rep}_{\mathbb{R}}(G) \xrightarrow{N_b} \mathrm{Isoc}_{\mathbb{R}} \xrightarrow{F} \mathrm{Rep}_{\mathbb{C}}(\mathbb{G}_m)$$

corresponds to

$$v_b : \mathbb{G}_m \to G_{\mathbb{C}}$$

and thus the result follows from the definition of a basic element.

Now assume b is basic., then $\mathrm{Rep}_{\mathbb{R}}(G)$ is endowed with a grading [DM18, Definition 5.1.]. This means for $(V, r) \in \mathrm{Rep}_{\mathbb{R}}(G)$, we have

$$(V, r) = \bigoplus_{n \in \mathbb{Z}} (V_n, r_n).$$

Furthermore $\omega(V_n, r_n)$ is a pure isocrystal of slope n. We define

$$\gamma : \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{GrBun}_X, \qquad (V, r) \mapsto \bigoplus_{n \in \mathbb{Z}} \mathcal{E}(\omega(V_n, r_n))$$

and this is easily seen to fulfill

$$\tilde{\omega} = \mathrm{fil} \circ \gamma$$

since $\mathcal{E}(\omega(V_n, r_n))$ is exactly the semistable subbundle of slope n. This functor is unique, since each representation decomposes uniquely into subrepresentations for which the functor is determined.

□

Chapter 6

$U(1)$-equivariant bundles on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$

In this chapter we will examine the role of $U(1)$-equivariant bundles on X. The main motivation is the following Theorem

Theorem. *[Sch18, Proposition 6.1] The category of $U(1)$-equivariant semistable vector bundles on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$ is equivalent to the category of pure $\mathbb{R}$-Hodge structures.*

We want to extend this to a triangle

$$\begin{array}{ccc} & \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}) & \\ \nearrow\!\!\!\swarrow & & \searrow \\ \mathrm{Isoc}_{\mathbb{R}}^{U(1)} & \longrightarrow & \mathrm{Bun}_X^{U(1)} . \end{array}$$

And in a more Tannakian style, for a reductive group G, this will draw a connection between $U(1)$-equivariant semistable G-bundles on X and certain functors $\mathrm{Hom}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}))$.

6.1 Equivariant sheaves and filtrations

The main interest of this section are equivariant sheaves. Our aim is to introduce the equivalences in Table 1.
In this section all vector spaces are finite dimensional.

To give the general definition of an equivariant sheaf let G be a group scheme over a scheme S and Z a scheme over S, with a G-action, i.e. a morphism

$$a : G \times_S Z \to Z$$

satisfying the group action axioms.

Definition 6.1.1. A G-equivariant quasi-coherent sheaf on Z is a quasi-coherent sheaf $\mathcal{F}$ on Z with an isomorphism of $\mathcal{O}_{G\times Z}$ modules

$$\theta : a^*\mathcal{F} \to \mathrm{pr}^*\mathcal{F}$$

such that

Quotient stack	Vector bundles on the quotient	Reference
$[\mathbb{G}_m/\mathbb{G}_m]$	Complex vector space V	Proposition 6.1.9.
$[\mathbb{A}^1_{\mathbb{C}}/\mathbb{G}_m]$	Complex vector space with a filtration (V, F^*V)	Lemma 6.1.8.
$[\mathbb{P}^1_{\mathbb{C}}/\mathbb{G}_m]$	Complex vector space with two filtrations $(V, F^pV, \bar{F}^mV)$	Lemma 6.1.16.
+semi-stability	Complex Hodge structure	Theorem 6.1.18.
$[(X-\infty)/U(1)]$	Real vector space V	Proposition 6.1.23.
$[X/U(1)]$	Real vector space with a filtration $(V, F^*V_{\mathbb{C}})$	Lemma 6.1.24.
+semi-stability	Real Hodge structure	Theorem 6.1.26.

Table 1

1. The diagram

$$\begin{array}{ccc} (1_G \times a)^* \mathrm{pr}_1^* \mathcal{F} & \xrightarrow[\mathrm{pr}_{12}^*\theta]{} & \mathrm{pr}_2^* \mathcal{F} \\ {\scriptstyle (1_G\times a)^*\theta}\uparrow & & \uparrow{\scriptstyle (m\times 1_X)^*\theta} \\ (1_G \times a)^* a^* \mathcal{F} & = & (m \times 1_X)^* a^* \mathcal{F} \end{array}$$

   commutes,

2. The pullback

$$(e \times 1_Z)^* \theta : \mathcal{F} \longrightarrow \mathcal{F}$$

   is the identity map.

Assume that $X = \mathrm{Spec}(A)$ is an affine scheme and $G = \mathrm{Spec}(B)$ is an affine group schemes over a field k, then we can give an alternative description of an equivariant sheaf.

Definition 6.1.2. A G-equivariant module M is an A-module M with a morphism

$$\gamma : M \to M \otimes_k B$$

satisfying the comodule axioms and making the action map

$$A \otimes_k M \to M$$

into a map of B comodules.

Lemma 6.1.3. *Let $X = \mathrm{Spec}(A)$ be an affine scheme and $G = \mathrm{Spec}(B)$ be affine group scheme over a field k. Then*

- *the category of G-equivariant modules and*
- *the category of G-equivariant quasi-coherent sheaves*

are equivalent.

Proof. [KR14, Proposition 3.2.] □

Example 6.1.4. Assume A is a graded ring over $\mathbb{C}$. Then we may define a morphism

$$a : A \to A \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}]$$

as the direct sum of the morphisms

$$a_n : A_n \to A_n \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}]$$
$$x \mapsto x \otimes T^n.$$

One can check that a is indeed a coaction, thus a grading on A induces an action of $\mathbb{G}_m$ on Spec(A). The converse is also true as the following Proposition shows.

Proposition 6.1.5. Let $a : \mathbb{G}_m \times X \to X$ be an action on an affine scheme. Then X is the spectrum of a $\mathbb{Z}$-graded ring and the action is as in the above example.

Proof. [Sta18, Tag 03LE]. □

Example 6.1.6. The affine line $\mathbb{A}^1_{\mathbb{C}} = Spec(\mathbb{C}[T])$ has a canonical grading, where $\deg(T^n) = n$. This corresponds to the action

$$\mathbb{G}_m \times \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$$
$$(t, x) \mapsto tx.$$

Lemma 6.1.7. *The category of $\mathbb{G}_m$-equivariant quasi-coherent sheaves on $\mathbb{A}^1_{\mathbb{C}}$ is equivalent to the category of $\mathbb{Z}$-graded $\mathbb{C}[T]$-modules.*

Proof. [Sta18, Tag 03LE]. □

Lemma 6.1.8. *The category of $\mathbb{G}_m$-equivariant vector bundles on $\mathbb{A}^1_{\mathbb{C}}$ is equivalent to the category of filtered vector spaces.*

Proof. We will only sketch the constructions, a full proof can be found in [Sim96, Section 5].

Let $\mathcal{V} = \widetilde{B}$ be a $\mathbb{G}_m$-equivariant vector bundle. We define for $m = (T-1)$ the vector space $V := B/mB = B_m/mB_m$. This comes with the morphism

$$q : B \to V.$$

As B is a $\mathbb{G}_m$-equivariant comodule, it comes with a grading

$$B = \bigoplus_{n \in \mathbb{Z}} B_n.$$

Now the filtration is defined as

$$F^pV = im(q : B_{-p} \to V). \tag{6.1}$$

Conversely start with a vector space and a filtration (V, F^*V), then the Rees module is defined as

$$\Lambda = \sum_{p \in \mathbb{Z}} z^{-p} F^pV \subset V \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}].$$

This inherits a grading from $V \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$, i.e. for $z^{-p}v \in \Lambda$ its degree is $-p$, which is compatible with the grading of $\mathbb{C}[z]$. □

In what follows we want to approach the above Lemma from a different angle, using the philosophy of Beauville-Laszlo's theorem. This will be essential, when we switch to the twistor case.

Proposition 6.1.9. The category of $\mathbb{G}_m$-equivariant vector bundles on $\mathbb{G}_m$ is equivalent to the category of vector spaces.

Proof. This is a general consequence of the fact that the action of $\mathbb{G}_m$ on itself is simply transitive.

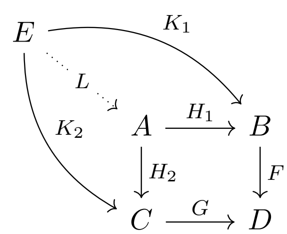
In this specific case we can even give an explicit isomorphism. Let $\mathcal{V} = \widetilde{B}$ be a $\mathbb{G}_m$-equivariant vector bundle. Then

$$B \to B/(T-1)B \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}]$$

sending an element $b \in B_n$ to $[b] \otimes T^n$ is a $\mathbb{G}_m$-equivariant isomorphism. $\square$

Heuristically speaking this means that we should be able to obtain every $\mathbb{G}_m$-equivariant bundle on $\mathbb{A}^1_{\mathbb{C}}$ by gluing a trivial $\mathbb{G}_m$-equivariant bundle on $\mathbb{G}_m$ and a $\mathbb{G}_m$-equivariant bundle on an infinitesimally small neighbourhood around $0 \in \mathbb{A}^1_{\mathbb{C}}$. The first step to making this notion precise is the Theorem by Beaville and Laszlo. Before introducing the theorem of interest, we need the notion of cartesian diagrams of categories.

Definition 6.1.10. Let A, B, C, D be categories with functors fitting into the following diagram



(6.2)

together with a natural isomorphism

$$\alpha : F \circ H_2 \to G \circ H_1.$$

Then the above diagram is cartesian, if it has the following universal property:

Given a category E with two functors K_1, K_2, as indicated above, and a natural isomorphism

$$\beta : G \circ K_2 \to F \circ K_1,$$

there exists an unique functor $L : E \to A$, such that

- $K_1 = H_1 \circ L$, $K_2 = H_2 \circ L$
- $\beta = \alpha \circ L$

Theorem 6.1.11. *Let A be a ring, f a non-zero divisor, and $\widehat{A}$ its (f)-adic completion. Let $M(A)$ be the category of a A-modules and $M_f(A)$ be the full subcategory of f-regular modules, i.e. modules, such that multiplication with f is injective on M. Then the following diagram of categories*

$$\begin{array}{ccc} M_f(A) & \longrightarrow & M_f(\widehat{A}) \\ \downarrow & & \downarrow \\ M(A_f) & \longrightarrow & M(\widehat{A}_f) \end{array}$$

is cartesian.

Proof. [BL95] □

Theorem 6.1.12. *Let Z be a smooth curve over an arbitrary base field k, p a closed point, and $Z^* := Z - \{p\}$. Let $D := \widehat{\mathcal{O}_{Z,p}}$ be the the infinitesimal neighbourhood around p, D^0 its fraction field, and $r \in \mathbb{N}$. Then the following diagram of categories*

$$\begin{array}{ccc} Bun^r(Z) & \longrightarrow & Bun^r(\mathrm{Spec}(D)) \\ \downarrow & & \downarrow \\ Bun^r(Z^*) & \longrightarrow & Bun^r(\mathrm{Spec}(D^0)) \end{array}$$

is cartesian.

Proof. This is just a global application of Theorem 6.1.11. □

Construction 6.1.13. More generally we can glue G-equivariant vector bundles on Z^* and D, if Z^* is stable under G. First notice that this problem is local, thus we can reduce the problem to the affine case, i.e. $Z = \mathrm{Spec}(A)$, $p = (f)$ for some $f \in A$ and thus $Z^* = \mathrm{Spec}(A_f)$.

Assume that we are given

- a G-equivariant projective A_f module M
- a G-equivariant projective D module Λ
- a G-equivariant isomorphism $\phi : M \otimes_{A_f} D^0 \to \Lambda \otimes_D D^0$,

then we may use Theorem 6.1.12. to glue the G-equivariant structure. First we glue the underlying modules by applying Theorem 6.1.12. to get a projective A-module N.

To glue the equivariant structures we want to apply Theorem 6.1.12. Consider $A \otimes \mathcal{O}(G)$ and the non zero-divisor $f \otimes 1$. Then we get a cartesian diagram

$$\begin{array}{ccc} M_f(A \otimes \mathcal{O}(G)) & \longrightarrow & M_f(\widehat{A} \otimes \mathcal{O}(G)) \\ \downarrow & & \downarrow F \\ M(A_f \otimes \mathcal{O}(G)) & \xrightarrow{G} & M(\widehat{A}_f \otimes \mathcal{O}(G)) \end{array}$$

It is sufficient to construct isomorphisms

$$\theta_f : a^*(N \otimes_A A_f) \to \mathrm{pr}^*(N \otimes_A A_f) \ , \ \theta_D : a^*(N \otimes_A D) \to \mathrm{pr}^*(N \otimes_A D)$$

in $M(A_f \otimes \mathcal{O}(G))$ and $M_f(\widehat{A} \otimes \mathcal{O}(G))$ such that the diagram

$$\begin{array}{ccc} G(a^*(N \otimes_A A_f)) & \xrightarrow{G(\theta_f)} & G(\mathrm{pr}^*(N \otimes_A A_f)) \\ \downarrow & & \downarrow \\ F(a^*(N \otimes_A D)) & \xrightarrow{G(\theta_D)} & F(\mathrm{pr}^*(N \otimes_A D)) \end{array}$$

commutes; here the vertical arrows are the ones obtained canonically from restriction. Thus we get an uniquely determined isomorphism

$$\theta : a^* N \to \mathrm{pr}^* N.$$

Obviously a G-equivariant module on A induces a datum as above. And as the above isomorphism is always uniquely determined by its datum, we have;

Proposition 6.1.14. In the above notation the following are equivalent

1. a G-equivariant A-module N
2. a datum (M, Λ, ϕ)

Now we may approach Lemma 6.1.8. with this new approach:

To give a $\mathbb{G}_m$-equivariant vector bundle on $\mathbb{A}^1_{\mathbb{C}}$ is equivalent to give

- a $\mathbb{G}_m$-equivariant vector bundle M on $\mathbb{G}_m$
- a $\mathbb{G}_m$-equivariant vector bundle Λ on $\mathrm{Spec}(D) = \mathrm{Spec}(\mathbb{C}[[z]])$
- a $\mathbb{G}_m$-equivariant isomorphism $\Lambda \otimes_D D^0 \xrightarrow{\sim} M \otimes_{\mathbb{C}[z,z^{-1}]} D^0$.

The crucial observation now is that every $\mathbb{G}_m$-equivariant vector bundle on $\mathbb{G}_m$ is trivial by Proposition 6.1.9. Thus we can summarize the above datum to giving

- a $\mathbb{G}_m$-equivariant lattice $\Lambda \subset V \otimes_{\mathbb{C}} \mathbb{C}((z))$.

These are of the form

$$\Lambda = \sum_{p \in \mathbb{Z}} z^{-p} F^p V[[z]]$$

and equivalent to a filtration of V. Going backwards we can construct a $\mathbb{G}_m$-equivariant vector bundle on $\mathbb{A}^1_{\mathbb{C}}$ from a filtration.

As the next step we also want to consider $\mathbb{G}_m$-equivariant vector bundles on $\mathbb{P}^1_{\mathbb{C}}$. There is a natural action of $\mathbb{G}_m$ on $\mathbb{P}^1_{\mathbb{C}}$:

$$\begin{aligned} a : \mathbb{G}_m \times \mathbb{P}^1_{\mathbb{C}} &\to \mathbb{P}^1_{\mathbb{C}} \\ (t, [a:b]) &\mapsto [ta : b]. \end{aligned}$$

Lemma 6.1.15. *The category of $\mathbb{G}_m$-equivariant vector bundles on $\mathbb{P}^1_{\mathbb{C}}$ is equivalent to the category of vector spaces with two filtrations.*

Proof. Again we sketch the constructions. Start with a $\mathbb{G}_m$-equivariant sheaf on $\mathbb{P}^1_{\mathbb{C}}$. By restriction we get

- a $\mathbb{G}_m$-equivariant sheaf on $\mathbb{A}^1_{\mathbb{C}}$
- a $\mathbb{G}_m$-equivariant sheaf on $\mathbb{P}^1_{\mathbb{C}} - \{0\}$.

These are, by Lemma 6.1.8., equivalent to two filtrations.

Conversely start with two filtrations $F^*V, \bar{F}^*V$. Again by lemma 6.1.8. these induce two vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ on $\mathbb{A}^1_{\mathbb{C}}$. By the inversion isomorphism $\mathbb{A}^1_{\mathbb{C}} \xrightarrow{\sim} \mathbb{P}^1_{\mathbb{C}} - \{0\}$, we may view $\mathcal{V}_2$ as a vector bundle on $\mathbb{P}^1_{\mathbb{C}} - \{0\}$. Now we glue these modules to get a vector bundle on $\mathbb{P}^1_{\mathbb{C}}$ as follows. By Proposition 6.1.9. we can choose $\mathbb{G}_m$-equivariant trivializations

$$\alpha_i : \mathcal{V}_i \mid_{\mathbb{G}_m} \xrightarrow{\sim} V \otimes \mathcal{O}_{\mathbb{G}_m}.$$

The isomorphism

$$\mathcal{V}_1 \xrightarrow{\alpha_1} V \otimes \mathcal{O}_{\mathbb{G}_m} \xrightarrow{id \otimes z^{-1}} V \otimes \mathcal{O}_{\mathbb{G}_m} \xrightarrow{\alpha_2^{-1}} \mathcal{V}_2$$

is $\mathbb{G}_m$-equivariant and thus we can glue these to get the desired $\mathbb{G}_m$-equivariant vector bundle on $\mathbb{P}^1_{\mathbb{C}}$. □

Remark 6.1.16. Again one may alternatively argue by Beauville-Laszlo. One first extends the trivial $\mathbb{G}_m$-equivariant sheaf on $\mathbb{G}_m$ to $\mathbb{A}^1_{\mathbb{C}}$ via the first filtration, then one extends it from $\mathbb{A}^1_{\mathbb{C}}$ to $\mathbb{P}^1_{\mathbb{C}}$ by extending it locally from $\mathbb{G}_m$ to $\mathbb{P}^1_{\mathbb{C}} - \{0\}$ via the second filtration. The resulting $\mathbb{G}_m$-equivariant sheaf is isomorphic to the one constructed in the proof above.

It seems natural to ask whether certain conditions on the filtration lead to certain $\mathbb{G}_m$-equivariant vector bundles.

Let V be a complex vector space with two filtrations $F^*V, \bar{F}^*V$ and define

$$H^{p,q} := F^pV \cap \bar{F}^qV.$$

We say that the two filtrations define a pure Hodge structure of weight n, if the subspaces $H^{p,q} \subset V$ define a pure complex Hodge structure of weight n on V, i.e.

$$V = \bigoplus_{p+q=n} H^{p,q}.$$

Theorem 6.1.17. *Two filtrations define a Hodge structure if and only if the corresponding $\mathbb{G}_m$-equivariant vector bundle on $\mathbb{P}^1_{\mathbb{C}}$ is semistable.*

Proof. [Sim96, Section 5] □

Remark 6.1.18. Here a $\mathbb{G}_m$-equivariant vector bundle is semistable, if the underlying vector bundle is.

Now after deriving this Theorem in the complex case, we would like to use descent to get real Hodge structures from certain equivariant vector bundles. As to be expected this is done by considering semistable vector bundles on X. We first construct a group action on X.

Construction 6.1.19. The following morphism

$$\begin{aligned} g : \mathbb{G}_m \times \mathbb{P}^1_{\mathbb{C}} &\to \mathbb{G}_m \times \mathbb{P}^1_{\mathbb{C}} \\ (t, [a : b]) &\mapsto (\frac{1}{\bar{t}}, [-\bar{b} : \bar{a}]). \end{aligned}$$

is a descent datum for $\mathbb{G}_m \times \mathbb{P}^1_{\mathbb{C}}$ and the descent is given by $U(1) \times X$ (as $\mathbb{G}_m$ descends to $U(1)$ via the morphism $\sigma_{U(1)} : t \mapsto \frac{1}{\bar{t}}$).

It is then straightforward to check that the diagram

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{P}^1_{\mathbb{C}} & \xrightarrow{a} & \mathbb{P}^1_{\mathbb{C}} \\ \downarrow g & & \downarrow f \\ \mathbb{G}_m \times \mathbb{P}^1_{\mathbb{C}} & \xrightarrow{a} & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

is commutative. Thus it descends to an action

$$U(1) \times X \xrightarrow{[a]} X.$$

Lemma 6.1.20. *The $U(1)$-action on $X^* = X - \{\infty\}$ is simply transitive.*

Proof. We want to show that

$$\begin{aligned} U(1) \times X^* &\to X^* \times X^* \\ (u, x) &\mapsto ([a](u, x), x) \end{aligned}$$

is an isomorphism. Going back to the complex case, we obviously have a simply transitive action of $\mathbb{G}_m$ on $\mathbb{P}^1_{\mathbb{C}} - \{0, \infty\}$, i.e.

$$\begin{aligned} \mathbb{G}_m \times (\mathbb{P}^1_{\mathbb{C}} - \{0, \infty\}) &\to (\mathbb{P}^1_{\mathbb{C}} - \{0, \infty\}) \times (\mathbb{P}^1_{\mathbb{C}} - \{0, \infty\}) \\ (\lambda, z) &\mapsto (a(\lambda, z), z). \end{aligned}$$

is an isomorphism. If we endow the left hand side with the descent datum $\sigma_U \times f$ and the right hand side with $f \times f$, the above isomorphism descends. Since $(\mathbb{P}^1_{\mathbb{C}} - \{0, \infty\}, f \mid_{\mathbb{P}^1_{\mathbb{C}} - \{0,\infty\}})$ descends to X^*, we get the desired isomorphism. □

Now we investigate the neighbourhood around $\infty \in X$:

Lemma 6.1.21. *The infinitesimal neighbourhood $D = \widehat{\mathcal{O}_{X,\infty}}$ around ∞ admits an isomorphism*

$$\rho : \mathbb{C}[[t]] \to D.$$

Proof. Since the local ring $(\mathcal{O}_{X,\infty}, m_\infty)$ is noetherian, regular, of dimension one, with residue field $\mathbb{C}$, so will be its completion $(\widehat{\mathcal{O}_{X,\infty}}, \widehat{m_\infty})$. But by Cohen's structure theorem [Sta18, Tag 0C0S] this already implies that there exists an isomorphism

$$\rho : \mathbb{C}[[t]] \to D.$$

□

We can even do better and give an explicit realization of the isomorphism ρ.

As $(D, \widehat{m_\infty})$ is a complete local ring, it is Henselian. Thus we can find a solution for the polynomial $f(X) = X^2 + 1 \in D[X]$ as we can find a solution inside

$$D/\widehat{m_\infty} \cong \mathcal{O}_{X,\infty}/m_\infty \cong \mathbb{C}$$

and obviously $f' \neq 0$ inside $D/\widehat{m_\infty}$. This yields an injection

$$\mathbb{C} \to D.$$

We can upgrade this injection by sending t to a, which is well-defined, as D is (a)-adically complete, to get an injection

$$\rho : \mathbb{C}[[t]] \to D.$$

Finally this is surjective by [Sta18, Tag 0315 (1)].

One sees that the geometric picture is quite similar to the complex case. The $U(1)$-action has one fixed point $\infty \in X$ and acts simply transitive on the complement $X - \{\infty\}$. Thus the preceding ideas, as explained in e.g. Proposition 6.1.14., apply here.

Proposition 6.1.22. The category of $U(1)$-equivariant vector bundles on X^* is equivalent to the category of real vector spaces.

Proof. As the action is simply transitive, $[X^*/U(1)]$ is represented by $\mathrm{Spec}(\mathbb{R})$ and the claim follows. □

Lemma 6.1.23. *The category of $U(1)$-equivariant vector bundles on X is equivalent to the category of real vector space with a filtration on its complexification.*

Proof. Here we apply Theorem 6.1.12. Thus an $U(1)$-equivariant vector bundle on X is equivalent to

- an $U(1)$-equivariant vector bundle M on X^*
- an $U(1)$-equivariant vector bundle Λ on $\mathrm{Spec}(D) = \mathrm{Spec}(\mathbb{C}[[z]])$
- an $U(1)$-equivariant isomorphism $\Lambda \otimes_D D^0 \xrightarrow{\sim} M \otimes_{\mathcal{O}_X(X^*)} D^0$.

But by Lemma 6.1.20. $U(1)$-equivariant vector bundles on X^* are always trivial, i.e. we can find an $U(1)$-equivariant isomorphism

$$M \cong V \otimes_{\mathbb{R}} \mathcal{O}_X(X^*).$$

Thus the above datum reduces to giving

- an $U(1)$-equivariant lattice $\Lambda \subset V \otimes_R \mathbb{C}(\!(z)\!) \cong V_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}(\!(z)\!)$.

As we already have seen before, this is equivalent to a filtration on $V_{\mathbb{C}}$. □

Remark 6.1.24. Thus the above lemma can be seen as a descent version of the prior case.

Finally, it is natural to ask when the pair $(V, F^* V_{\mathbb{C}})$ yields a pure real Hodge structure.

Theorem 6.1.25. *The category of pure Hodge structures is equivalent to the category of $U(1)$-equivariant semistable vector bundles on X.*

Proof. [Sch18, Proposition 6.1.] □

The functor

$$\mathcal{R} : \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}) \to \mathrm{Bun}_X^{U(1)}$$

inducing the equivalence sends a pure Hodge structure $(V, V^{p,q})$ to the $U(1)$-equivariant lattice

$$\Lambda = \sum_{p \in \mathbb{Z}} t^{-p} (F^p_{Hod} V)[[t]]$$

and an application of Lemma 6.1.23. yields the desired $U(1)$-equivariant vector bundle. Finally we can extend this construction to direct sums of pure Hodge structures.

6.2 $U(1)$-equivariant isocrystals

In Chapter 2 we presented an ad-hoc definition of $\mathrm{Isoc}_{\mathbb{R}}^{U(1)}$, which in this section will be motivated. Our aim in this section is to construct a functor

$$\mathcal{E}^{U(1)} : \mathrm{Isoc}_{\mathbb{R}}^{U(1)} \to \mathrm{Bun}_X^{U(1)}.$$

This functor should of course be compatible with our functor $\mathcal{E} : \mathrm{Isoc}_{\mathbb{R}} \to \mathrm{Bun}_X$. Furthermore we will show that this functor is compatible with the construction in Theorem 6.1.25., in the sense that (up to natural isomorphism) the following diagram commutes

$$\begin{array}{ccccc} & & \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}) & & \\ & \overset{\mathcal{H}}{\nearrow\!\!\swarrow} & & \overset{\mathcal{R}}{\searrow} & \\ \mathrm{Isoc}_{\mathbb{R}}^{U(1)} & & \xrightarrow{\quad \mathcal{E}^{U(1)} \quad} & & \mathrm{Bun}_X^{U(1)} \end{array}.$$

Recall the definition of the category $\mathrm{Isoc}_{\mathbb{R}}^{U(1)}$.

Definition 6.2.1.
- The objects are isocrystals $(V, \phi) \in \mathrm{Ob}(\mathrm{Isoc}_{\mathbb{R}})$ with a comodule map $\gamma_V : V \to V \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}]$ respecting the grading, i.e. $\gamma_V(V_n) \subset V_n \otimes \mathbb{C}[T, T^{-1}]$, such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\gamma_V} & V \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}] \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi \otimes \sigma_V} \\ V & \xrightarrow{\gamma_V} & V \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}]. \end{array}$$

- A homomorphism between two objects is a homomorphism $f \in \mathrm{Hom}_{\mathrm{Isoc}_{\mathbb{R}}}(V, W)$, such that

$$\begin{array}{ccc} V & \xrightarrow{\gamma_V} & V \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \otimes id} \\ W & \xrightarrow{\gamma_W} & W \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}] \end{array}$$

  commutes.

Construction 6.2.2. Choose an object (V, ϕ, γ) in $\mathrm{Isoc}_{\mathbb{R}}^{U(1)}$ of slope n. Recall that we have an equivalence

$$\mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X) \xrightarrow{\sim} \mathrm{Bun}_X .$$

In the last chapter we constructed an object $(\mathcal{E}(V), v : f^*\mathcal{E}(V) \to \mathcal{E}(V)) \in \mathrm{Ob}(\mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X))$; write $[\mathcal{E}(V)]$ for its image in Bun_X. We want to obtain an isomorphism

$$\widetilde{\theta} : [a]^*[\mathcal{E}(V)] \xrightarrow{\sim} [\mathrm{pr}_X]^*[\mathcal{E}(V)].$$

Assume that we have a commutative diagram of the form

$$\begin{array}{ccc} a^*\mathcal{E}(V) & \xrightarrow{\theta} & \mathrm{pr}^* \mathcal{E}(V) \\ {\scriptstyle a^*v}\uparrow & & {\scriptstyle \mathrm{pr}^* v}\uparrow \\ g^*a^*\mathcal{E}(V) & \xrightarrow{g^*\theta} & g^* \mathrm{pr}^* \mathcal{E}(V) \end{array} \tag{6.3}$$

where by abuse of notation we identified $g^*a^*\mathcal{E}(V) \cong a^*f^*\mathcal{E}(V)$ and $g^* \mathrm{pr}^* \mathcal{E}(V) \cong \mathrm{pr}^* f^*\mathcal{E}(V)$. Then

- the pair $(a^*\mathcal{E}(V), a^*v : g^*a^*\mathcal{E}(V) \to a^*\mathcal{E}(V))$ descends to an object $[a^*\mathcal{E}(V)]$, which can be identified with $[a^*][\mathcal{E}(V)]$ (resp.for $\mathrm{pr}^* \mathcal{E}(V)$)

- the morphism $\theta : a^*\mathcal{E}(V) \to \mathrm{pr}^*\,\mathcal{E}(V)$ descends to a morphism $[\theta] : [a^*\mathcal{E}(V)] \to [\mathrm{pr}^*\,\mathcal{E}(V)]$.

The second point is a straightforward consequence of descent of morphisms. And up to the above identification, this yields the required $\widetilde{\theta}$.

Proposition 6.2.3. In the above construction we can identify

$$[a^*][\mathcal{E}(V)] \xrightarrow{\sim} [a^*\mathcal{E}(V)].$$

Proof. First reduce this problem to the affine case, as the statement is local. The statement follows then from the following general isomorphism:

Assume we have $\mathbb{C}$-algebras A, B, such that A is an B-module. Furthermore assume that these have real forms denoted by $[A], [B], [M]$. Then

$$M \otimes_B A \cong ([M] \otimes \mathbb{C}) \otimes_{[B]\otimes\mathbb{C}} A \cong [M] \otimes_{[B]} A \cong [M] \otimes_{[B]} [A] \otimes \mathbb{C}$$

and this isomorphism preserves the descent data, i.e. the antiholomorphic involutions . □

Thus we will be interested in constructing diagrams of the form (6.3). We now describe how to construct these out of objects in $\mathrm{Isoc}_{\mathbb{R}}^{U(1)}$.

Construction 6.2.4. We will construct an isomorphism

$$\theta_i : a^*\mathcal{E}(V)\mid_{W_i} \to \mathrm{pr}^*\,\mathcal{E}(V)\mid_{W_i}$$

on the affine opens $W_1 = \mathbb{A}^1_{\mathbb{C}}$ and $W_2 = \mathbb{P}^1_{\mathbb{C}} - \{\infty\}$. They will agree on the intersection, thus we can glue these to a global isomorphism on $\mathbb{G}_m \times \mathbb{P}^1_{\mathbb{C}}$. We first define a $\mathbb{G}_m$-equivariant structure of $\mathcal{E}(V)$ on $\mathbb{A}^1_{\mathbb{C}}$.

The comodule structure on V_n induces a grading $V_n = \bigoplus_{m\in\mathbb{Z}} V_n^m$. For homogeneous elements $w \in V_n$ define

$$\theta_1 : \big(V_n \otimes \frac{1}{z^n}\mathbb{C}[z]\big) \otimes_{a,\mathbb{C}[z]} (\mathbb{C}[z] \otimes \mathbb{C}[T, T^{-1}]) \to \big(V_n \otimes \frac{1}{z^n}\mathbb{C}[z]\big) \otimes_{pr,\mathbb{C}[z]} (\mathbb{C}[z] \otimes \mathbb{C}[T, T^{-1}])$$
$$w \otimes z^h \otimes 1 \otimes 1 \mapsto w \otimes z^h \otimes 1 \otimes T^{\deg(w)+h}.$$

This is easily generalized to a well-defined morphism between $\mathbb{C}[z] \otimes \mathbb{C}[T, T^{-1}]$-modules. Let us now evaluate diagram (6.3) at W_1

$$\begin{array}{ccc}
(V \otimes \frac{1}{z^n}\mathbb{C}[z]) \otimes_{\mathbb{C}[z],a} (\mathbb{C}[z] \otimes \mathbb{C}[T, T^{-1}]) & \xrightarrow{\theta_1} & (V \otimes \frac{1}{z^n}\mathbb{C}[z]) \otimes_{\mathbb{C}[z],pr} (\mathbb{C}[z] \otimes \mathbb{C}[T, T^{-1}]) \\
\uparrow{\scriptstyle a^*v} & & \uparrow{\scriptstyle \mathrm{pr}^*\,v} \\
(V \otimes \mathbb{C}[\frac{1}{z}]) \otimes_{\mathbb{C}[\frac{1}{z}],a\circ g} (\mathbb{C}[z] \otimes \mathbb{C}[T, T^{-1}]) & \xrightarrow{g^*\theta_1} & (V \otimes \mathbb{C}[\frac{1}{z}]) \otimes_{\mathbb{C}[\frac{1}{z}],pr\circ g} (\mathbb{C}[z] \otimes \mathbb{C}[T, T^{-1}]).
\end{array}$$

Now we describe the individual maps

$$a^*v : (V\otimes\mathbb{C}[\frac{1}{z}])\otimes_{\mathbb{C}[\frac{1}{z}],a\circ g}(\mathbb{C}[z]\otimes\mathbb{C}[T,T^{-1}])\to(V\otimes\frac{1}{z^n}\mathbb{C}[z])\otimes_{\mathbb{C}[z],a}(\mathbb{C}[z]\otimes\mathbb{C}[T,T^{-1}])$$
$$w\otimes p(\frac{1}{z})\otimes 1\otimes 1\mapsto \phi(w)\otimes\bar{p}(-z)\cdot\frac{1}{z^n}\otimes 1\otimes 1$$
$$\operatorname{pr}^* v : (V\otimes\mathbb{C}[\frac{1}{z}])\otimes_{\mathbb{C}[\frac{1}{z}],pr\circ g}(\mathbb{C}[z]\otimes\mathbb{C}[T,T^{-1}])\to(V\otimes\frac{1}{z^n}\mathbb{C}[z])\otimes_{\mathbb{C}[z],pr}(\mathbb{C}[z]\otimes\mathbb{C}[T,T^{-1}])$$
$$w\otimes p(\frac{1}{z})\otimes 1\otimes 1\mapsto \phi(w)\otimes\bar{p}(-z)\cdot\frac{1}{z^n}\otimes 1\otimes 1$$
$$g^*\theta_1 : (V\otimes\mathbb{C}[\frac{1}{z}])\otimes_{\mathbb{C}[\frac{1}{z}],a\circ g}(\mathbb{C}[z]\otimes\mathbb{C}[T,T^{-1}])\to(V\otimes\mathbb{C}[\frac{1}{z}])\otimes_{\mathbb{C}[\frac{1}{z}],pr\circ g}(\mathbb{C}[z]\otimes\mathbb{C}[T,T^{-1}])$$
$$w\otimes z^m\otimes 1\otimes 1\mapsto v\otimes z^m\otimes 1\otimes T^{-(\deg(w)+m)}.$$

And we have

Proposition 6.2.5. The above diagram is commutative.

This is a brute force calculation and the essential point is the following: For a homogeneous $w\in V$ we have in the one direction

$$\theta_1(a^*v(w\otimes 1\otimes 1\otimes 1))=\theta_1(\phi(w)\otimes\frac{1}{z^n}\otimes 1\otimes 1)=\phi(w)\otimes\frac{1}{z^n}\otimes 1\otimes T^{\deg(\phi(w))-n}$$

and in the other

$$\operatorname{pr}^* v(g^*\theta_1(w\otimes 1\otimes 1\otimes 1))=\operatorname{pr}^* v(w\otimes 1\otimes 1\otimes T^{-\deg(w)})=\phi(w)\otimes\frac{1}{z^n}\otimes 1\otimes T^{-\deg(w)}.$$

And these two are equal, since by definition

$$\phi(V^{\deg(w)})\subset V^{n-\deg(w)}.$$

In a similar way we can define the diagram on the other affine open W_2 and a similar calculation shows again that the diagram will be commutative.

We summarize the result of the above constructions;

Lemma 6.2.6. *There exists a functor*

$$\mathcal{E}^{U(1)} : \operatorname{Isoc}_{\mathbb{R}}^{U(1)}\to\operatorname{Bun}_X^{U(1)}.$$

making the following diagram commutative

$$\begin{array}{ccc}\operatorname{Isoc}_{\mathbb{R}}^{U(1)} & \xrightarrow{\mathcal{E}^{U(1)}} & \operatorname{Bun}_X^{U(1)}\\ \downarrow{\scriptstyle Forget} & & \downarrow{\scriptstyle Forget}\\ \operatorname{Isoc}_{\mathbb{R}} & \xrightarrow{\mathcal{E}} & \operatorname{Bun}_X .\end{array}$$

Proof. We first treat the case of pure isocrystals by combining the previous two construction.

Construction 6.2.4. gives a commutative diagram

$$\begin{array}{ccc}a^*\mathcal{E}(V) & \xrightarrow{\theta} & \operatorname{pr}^*\mathcal{E}(V)\\ {\scriptstyle a^*v}\uparrow & & {\scriptstyle \operatorname{pr}^* v}\uparrow\\ g^*a^*\mathcal{E}(V) & \xrightarrow{g^*\theta} & g^*\operatorname{pr}^*\mathcal{E}(V)\end{array}$$

which by Construction 6.2.2. descends to an $U(1)$-equivariant vector bundle.

For functoriality one checks that a morphism f between two $U(1)$-equivariant isocrystals induces a $\mathbb{G}_m$-equivariant morphism of the underlying vector bundles in $\mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X)$, which then descends to an $U(1)$-equivariant morphism in Bun_X.

Finally, the above constructions are compatible with direct sums and thus we can generalize to the case of general isocrystals. $\square$

As announced above, we will now show that the following diagram commutes.

$$\begin{array}{ccc} & \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}) & \\ {\scriptstyle H}\nearrow\!\!\swarrow & & \searrow{\scriptstyle \mathcal{R}} \\ \mathrm{Isoc}_{\mathbb{R}}^{U(1)} & \xrightarrow{\quad\mathcal{E}^{U(1)}\quad} & \mathrm{Bun}_X^{U(1)}\,. \end{array}$$

The strategy is to compare the resulting $U(1)$-equivariant vector bundles as $\mathbb{G}_m$-equivariant sheaves on $\mathbb{P}^1_{\mathbb{C}}$ with their descent data. In particular Lemma 6.1.15. will be of use here.

Throughout this section let (V, ϕ, γ) be a pure $U(1)$-equivariant isocrystal of slope n, i.e. a pure isocrystal endowed with a comodule structure $\gamma : V \to V \otimes \mathbb{C}[T, T^{-1}]$ or equivalently a decomposition

$$V = \bigoplus_{m \in \mathbb{Z}} V^m$$

which fulfills

$$\phi(V^m) \subset V^{n-m}.$$

We begin by inspecting the filtrations induced by $\mathcal{E}^{U(1)}(H(V, V^{p,q}))$.

Lemma 6.2.7. *The $\mathbb{G}_m$-equivariant sheaf $\mathcal{E}^{U(1)}(H(V, V^{p,q}))$ corresponds*

- *to the filtration $F^p_{Hod}V = \bigoplus_{i \geq p} V^{i,n-i}$ on $\mathbb{A}^1_{\mathbb{C}}$*
- *to the filtration $\overline{F^p_{Hod}V} = \bigoplus_{i \geq p} V^{n-i,i}$ on $\mathbb{P}^1_{\mathbb{C}} - \{\infty\}$.*

Proof. We start with the $\mathbb{A}^1_{\mathbb{C}}$ case. The equivariant structure gives a decomposition

$$V_{\mathbb{C}} \otimes \frac{1}{z^n}\mathbb{C}[z] = \bigoplus_{m \in \mathbb{Z}} B^m.$$

Furthermore we have the quotient map

$$q : V_{\mathbb{C}} \otimes \frac{1}{z^n}\mathbb{C}[z] \to V_{\mathbb{C}} \otimes (\frac{1}{z^n}\mathbb{C}[z])/(z-1) \cong V_{\mathbb{C}}$$

By Construction 6.2.4. we see that the degree m part is given by

$$B^m = \langle i \geq -n \mid z^i \cdot v^{n-(m-i),m-i} \rangle.$$

According to lemma 6.1.8., we have

$$F^p V_{\mathbb{C}} = \mathrm{im}(q : B^{-p} \to V_{\mathbb{C}}).$$

Thus

$$F^pV_{\mathbb{C}} = \langle i \geq -n \mid v^{n-(-p-i),-p-i)} \rangle = \langle i \geq 0 \mid v^{p+i,(n-p)-i)} \rangle = \bigoplus_{i \geq p} V^{i,n-i} = F^p_{Hod}V.$$

The $\mathbb{P}^1_{\mathbb{C}} - \{\infty\}$ case is treated similarly. In this case we have the quotient map

$$q : V_{\mathbb{C}} \otimes \mathbb{C}[\frac{1}{z}] \to V_{\mathbb{C}} \otimes \mathbb{C}[\frac{1}{z}]/(\frac{1}{z} - 1) \cong V_{\mathbb{C}}.$$

The degree m part is given by

$$B^m = \langle i \geq 0 \mid \frac{1}{z^i} \cdot v^{n-(m+i),m+i} \rangle.$$

Now the filtration is given by

$$F^pV_{\mathbb{C}} = \mathrm{im}(q : B^p \to V_{\mathbb{C}}).$$

Thus

$$F^pV_{\mathbb{C}} = \langle i \geq 0 \mid v^{n-(p+i),p+i} \rangle = \bigoplus_{i \geq p} V^{n-i,i} = \overline{F^p_{Hod}V}.$$

□

Now we turn to $\mathcal{R}(V, V^{p,q})$. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\widehat{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}},\infty}}) \coprod \mathrm{Spec}(\widehat{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}},0}}) & \longrightarrow & \mathbb{P}^1_{\mathbb{C}} \\ \downarrow{\scriptstyle \pi_0 \coprod \pi_\infty} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(\widehat{\mathcal{O}_{X,\infty}}) & \longrightarrow & X \end{array}$$

or more explicitly by Lemma 6.1.21.

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{C}[[t_0]]) \coprod \mathrm{Spec}(\mathbb{C}[[t_\infty]]) & \longrightarrow & \mathbb{P}^1_{\mathbb{C}} \\ \downarrow{\scriptstyle \pi_0 \coprod \pi_\infty} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(\mathbb{C}[[t]]) & \longrightarrow & X. \end{array}$$

Upon fixing a choice for the root of $f(x) = x^2 + 1$ at the residue field of $\infty \in X$, we find that $\pi_0 \coprod \pi_\infty = \mathrm{id} \coprod \sigma$, i.e. the second projection is the complex conjugation.

Lemma 6.2.8. *The $\mathbb{G}_m$-equivariant sheaf $\mathcal{R}(V, V^{p,q})$ corresponds*

- *to the filtration $F^p_{Hod}V = \bigoplus_{i \geq p} V^{i,n-i}$ on $\mathbb{A}^1_{\mathbb{C}}$*
- *to the filtration $\overline{F^p_{Hod}V} = \bigoplus_{i \geq p} V^{n-i,i}$ on $\mathbb{P}^1_{\mathbb{C}} - \{\infty\}$.*

Proof. Recall that the $U(1)$-equivariant structure is given via the lattice

$$\Lambda = \sum_{p \in \mathbb{Z}} t^{-p} F^p_{Hod}V[[t]]$$

and pulling back this lattice via $\pi_0 \coprod \pi_\infty$, we get two lattices at the infinitesimal neighbourhood around 0 and ∞:

$$\Lambda_0 = \sum_{p\in\mathbb{Z}} t_0^{-p} F^p_{Hod} V[[t_0]] \ , \ \Lambda_\infty = \sum_{p\in\mathbb{Z}} t_\infty^{-p} \overline{F^p_{Hod} V}[[t_\infty]].$$

The lattices are exactly the the restriction of the $\mathbb{G}_m$-equivariant vector bundle

$$\mathcal{R}(V, V^{p,q})$$

to 0 and ∞ by the above commutative diagram. □

Comparing the above two Lemmas, we see that $\mathcal{R}(V, V^{p,q})$ and $\mathcal{E}^{U(1)}(\mathcal{H}(V, V^{p,q}))$ yield the same filtrations. If we can find $\mathbb{G}_m$ isomorphisms

$$\mathcal{E}^{U(1)}(\mathcal{H}(V, V^{p,q})) \mid_{\mathbb{G}_m} \cong \mathcal{R}(V, V^{p,q}) \mid_{\mathbb{G}_m}$$
$$\mathcal{E}^{U(1)}(\mathcal{H}(V, V^{p,q})) \mid_{\mathrm{Spec}(\widehat{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}},\infty}})} \cong \mathcal{R}(V, V^{p,q}) \mid_{\mathrm{Spec}(\widehat{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}},\infty}})}$$
$$\mathcal{E}^{U(1)}(\mathcal{H}(V, V^{p,q})) \mid_{\mathrm{Spec}(\widehat{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}},0}})} \cong \mathcal{R}(V, V^{p,q}) \mid_{\mathrm{Spec}(\widehat{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}},0}})}$$

which glue together and respect the descent data (i.e. the complex conjugation), then we are done. The first isomorphism is given by

$$\beta : V_{\mathbb{C}} \otimes \mathbb{C}[T, T^{-1}] \xrightarrow{\sim} V_{\mathbb{C}} \otimes \mathbb{C}[T, T^{-1}]$$
$$v^{p,q} \otimes T^m \mapsto v^{p,q} \otimes T^{m+q}.$$

and one checks that this respects the descent datum.

On the lattice at 0 we can define the morphism β_0^* to make the following diagram commute

$$\begin{array}{ccc} V_{\mathbb{C}} \otimes \mathbb{C}((t_0)) & \xrightarrow{\beta_0} & V_{\mathbb{C}} \otimes \mathbb{C}((t_0)) \\ \uparrow & & \uparrow \\ V_{\mathbb{C}} \otimes \frac{1}{t_0^n}\mathbb{C}[[t_0]] & \xrightarrow{\beta_0^*} & \sum_{p\in\mathbb{Z}} t_0^{-p} F^p_{Hod} V[[t_0]]. \end{array}$$

Here β_0 is the morphism β restricted to the punctured infinitesimal neighbourhood. For $\infty \in \mathbb{P}^1_{\mathbb{C}}$ we get a similar diagram

$$\begin{array}{ccc} V_{\mathbb{C}} \otimes \mathbb{C}((t_\infty)) & \xrightarrow{\beta_\infty} & V_{\mathbb{C}} \otimes \mathbb{C}((t_\infty)) \\ \uparrow & & \uparrow \\ V_{\mathbb{C}} \otimes \mathbb{C}[[t_\infty]] & \xrightarrow{\beta_\infty^*} & \sum_{p\in\mathbb{Z}} t_\infty^{-p} \overline{F^p_{Hod} V}[[t_\infty]]. \end{array}$$

Proposition 6.2.9. The above maps glue together and commute with the descent data.

Proof. The maps glue together by the their construction, i.e. by the commutativity of the above diagrams.

Commutativity with the descent data can be checked locally, i.e. for β, β_0^*, and β_∞^*. For example

$$\begin{array}{ccc} V_{\mathbb{C}} \otimes \mathbb{C}[T, T^{-1}] & \xrightarrow{\quad\beta\quad} & V_{\mathbb{C}} \otimes \mathbb{C}[T, T^{-1}] \\ {\scriptstyle \phi\otimes\delta_n}\uparrow & & {\scriptstyle \sigma\otimes\delta_n}\uparrow \\ V_{\mathbb{C}} \otimes \mathbb{C}[T, T^{-1}] \otimes_{\mathbf{f}} \mathbb{C}[T, T^{-1}] & \xrightarrow{f^*\beta} & V_{\mathbb{C}} \otimes \mathbb{C}[T, T^{-1}] \otimes_{\mathbf{f}} \mathbb{C}[T, T^{-1}] \end{array}$$

commutes, if σ denotes the complex conjugation on $V_{\mathbb{C}}$, since $\phi = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \circ \sigma$ (cf. Construction 2.1.11.). The other cases are treated similarly. □

6.3 Descent via semistable G-bundles

Definition 6.3.1. Let $\mathcal{V}$ be a vector bundle on X with an $U(1)$-equivariant structure. A subbundle

$$i : \mathcal{W} \hookrightarrow \mathcal{V}$$

is $U(1)$-equivariant, if $\mathcal{W}$ has as $U(1)$-equivariant structure, such that the inclusion is $U(1)$-equivariant.

Remark 6.3.2. Equivalently we can require that in the diagram

$$\begin{array}{ccc} a^*\mathcal{W} & \xhookrightarrow{a^*i} & a^*\mathcal{V} \\ \vdots & & \downarrow{\scriptstyle \theta} \\ \operatorname{pr}^*\mathcal{W} & \xhookrightarrow{\operatorname{pr}^* i} & \operatorname{pr}^*\mathcal{V} \end{array}$$

the morphism $\theta \circ a^*i : a^*\mathcal{W} \to \operatorname{pr}^*\mathcal{V}$ factors through $\operatorname{pr}^*\mathcal{W}$.

Definition 6.3.3. The category of $U(1)$-equivariant graded vector bundles $\operatorname{GrBun}_X^{U(1)}$ is the category of graded vector bundles with an $U(1)$-equivariant structure on each direct summand.

Definition 6.3.4. The category of $U(1)$-equivariant filtered vector bundles $\operatorname{FilBun}_X^{U(1)}$ is the category of filtered vector bundles with an $U(1)$-equivariant structure on the $F^n\mathcal{V}$ that respects the inclusion $F^{n+1}\mathcal{V} \subset F^n\mathcal{V}$.

Now we want to construct $U(1)$-equivariant versions of the functors we encountered in Section 5.2.

Construction 6.3.5. First recall the functor

$$\operatorname{gr} : \operatorname{FilBun}_X \to \operatorname{GrBun}_X \ , \ F^n\mathcal{V} \mapsto \bigoplus_{n\in\mathbb{Z}} F^n\mathcal{V}/F^{n+1}\mathcal{V}.$$

There is a straightforward way to define a functor

$$\operatorname{gr}^{U(1)} : \operatorname{FilBun}_X^{U(1)} \to \operatorname{GrBun}_X^{U(1)} .$$

Namely if $\mathcal{W} \subset \mathcal{V}$ is a $U(1)$-equivariant subsheaf, then

$$\begin{array}{ccccc} a^*\mathcal{W} & \xhookrightarrow{a^*i} & a^*\mathcal{V} & \longrightarrow & a^*\mathcal{V}/a^*\mathcal{W} \\ \downarrow & & \downarrow & & \vdots \\ \operatorname{pr}^*\mathcal{W} & \xhookrightarrow{\operatorname{pr}^* i} & \operatorname{pr}^*\mathcal{V} & \longrightarrow & \operatorname{pr}^*\mathcal{V}/\operatorname{pr}^*\mathcal{W} \end{array}$$

endows $\mathcal{V}/\mathcal{W}$ with an $U(1)$-equivariant structure. Here we use that for quasi-coherent sheaves $a^*(\mathcal{V}/\mathcal{W}) \cong a^*\mathcal{V}/a^*\mathcal{W}$ (respectively for pr^*). We define $\mathrm{gr}^{U(1)}$ to be gr on the underlying vector bundles and by the above we can endow each direct summand with an $U(1)$-equivariant structure.

Lemma 6.3.6. *Let $\mathcal{V}$ be an $U(1)$-equivariant vector bundle. The Harder-Narasimhan filtration of the underlying vector bundle*

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset ... \subset \mathcal{V}_n = \mathcal{V}$$

is $U(1)$-equivariant, i.e. $\mathcal{V}_i \subset \mathcal{V}$ is an $U(1)$-equivariant subbundle.

Proof. We will show that the Harder-Narasimhan filtration on $\mathbb{P}^1_{\mathbb{C}}$ is $\mathbb{G}_m$-equivariant. The above result is then a consequence of descent.

First of all we check that this statement is true for $\mathbb{G}_m(\mathbb{C})$, i.e. we have an action

$$\mathbb{G}_m(\mathbb{C}) \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$$

which on rings is induced by the isomorphism

$$\lambda^* : \mathbb{C}[a, b] \to \mathbb{C}[a, b]$$

which sends a to λa. In this case $\lambda^*\mathcal{W} \cong \mathcal{W}$ and λ^* preserves semistable bundles. Thus it also preserves the Harder-Narasimhan filtration.

The general case is deduced since $\mathbb{G}_m(\mathbb{C})$ is dense inside $\mathbb{G}_m$ and being an $U(1)$-equivariant subbundle is equivalent to the condition that two subbundles $\theta(a^*i(a^*W)) = \mathrm{pr}^*\, i(\mathrm{pr}^*\,\mathcal{W})$ agree. $\square$

Proposition 6.3.7. There exists a functor

$$\mathrm{HN}^{U(1)} : \mathrm{Bun}_X^{U(1)} \to \mathrm{FilBun}_X^{U(1)}$$

such that

$$\begin{array}{ccc} \mathrm{Bun}_X^{U(1)} & \xrightarrow{\mathrm{HN}^{U(1)}} & \mathrm{FilBun}_X^{U(1)} \\ {\scriptstyle Forget}\downarrow & & \downarrow{\scriptstyle Forget} \\ \mathrm{Bun}_X & \xrightarrow{\mathrm{HN}} & \mathrm{FilBun}_X \end{array}$$

commutes.

Proof. This is a consequence of the above lemma. $\square$

Lemma 6.3.8. *The composite functor*

$$\mathrm{Isoc}_{\mathbb{R}}^{U(1)} \xrightarrow{\mathcal{E}^{U(1)}} \mathrm{Bun}_X^{U(1)} \xrightarrow{\mathrm{HN}^{U(1)}} \mathrm{FilBun}_X^{U(1)} \xrightarrow{\mathrm{gr}^{U(1)}} \mathrm{GrBun}_X^{U(1)}$$

is an equivalence of exact categories from $\mathrm{Isoc}_{\mathbb{R}}$ *to its essential image in* GrBun_X*, which consists of graded vector bundles*

$$(\mathcal{E}, \theta) = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} (\mathcal{E}_i, \theta_i)$$

such that $\mathcal{E}_i$ is semistable of slope i.

Proof. We have seen that $\mathcal{E}^{U(1)} \cong \mathcal{R} \circ \mathcal{H}$. By Theorem 6.1.25. $\mathcal{R}$ is an equivalence when restricted to pure Hodge structures, i.e. $\mathcal{E}^{U(1)}$ is an equivalence when restricted to pure isocrystals. □

Theorem 6.3.9. *The bijection*

$$\{\textit{basic elements } \operatorname{Hom}(\operatorname{Rep}_{\mathbb{R}}(G), \operatorname{Isoc}_{\mathbb{R}})\} \longleftrightarrow \{\textit{semistable } G\textit{-bundles }\}/\cong$$

from Theorem 5.2.7. admits an $U(1)$*-equivariant version*

$$\{\textit{basic elements } \operatorname{Hom}(\operatorname{Rep}_{\mathbb{R}}(G), \operatorname{Isoc}_{\mathbb{R}}^{U(1)})\} \longleftrightarrow \{\textit{semistable } U(1)\textit{-equivariant } G\textit{-bundles}\}/\cong$$

Proof. Injectivity follows from Lemma 6.3.8.

To prove surjectivity, we use Theorem 5.2.7. Let

$$\omega : \operatorname{Rep}_{\mathbb{R}}(G) \to \operatorname{Bun}_X^{U(1)}$$

be semistable, i.e. composing with the forgetful functor

$$\operatorname{For} : \operatorname{Bun}_X^{U(1)} \to \operatorname{Bun}_X$$

yields a semistable bundle. Now for

$$\operatorname{HN} \circ \operatorname{For} \circ\, \omega : \operatorname{Rep}_{\mathbb{R}}(G) \to \operatorname{FilBun}_X$$

there is a unique splitting

$$\gamma : \operatorname{Rep}_{\mathbb{R}}(G) \to \operatorname{GrBun}_X .$$

In particular there is a natural isomorphism

$$\alpha : \gamma \to \operatorname{gr} \circ \operatorname{HN} \circ \operatorname{For} \circ\, \omega.$$

By the prior results we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Bun}_X^{U(1)} & \xrightarrow{\operatorname{For}} & \operatorname{Bun}_X \\ \Big\downarrow{\scriptstyle \operatorname{gr}^{U(1)} \circ \operatorname{HN}^{U(1)}} & & \Big\downarrow{\scriptstyle \operatorname{gr} \circ \operatorname{HN}} \\ \operatorname{GrBun}_X^{U(1)} & \xrightarrow{\operatorname{For}} & \operatorname{GrBun}_X \end{array} .$$

This means for any object $(V, r) \in \operatorname{Rep}_{\mathbb{R}}(G)$ we can endow $(\operatorname{gr} \circ \operatorname{HN} \circ \operatorname{For} \circ\, \omega)(V, r)$ with an $U(1)$-structure and, via the isomorphism α, we can also endow $\gamma(V, r)$ with an $U(1)$-structure, i.e. we get a functor

$$\gamma^{U(1)} : \operatorname{Rep}_{\mathbb{R}}(G) \to \operatorname{GrBun}_X^{U(1)}$$

and this comes with a natural isomorphism

$$\alpha^{U(1)} : \gamma^{U(1)} \to \operatorname{gr}^{U(1)} \circ \operatorname{HN}^{U(1)} \circ \omega.$$

In the spirit of Theorem 5.2.18. we can now define

$$\omega' : \operatorname{Rep}_{\mathbb{R}}(G) \to \operatorname{Isoc}_{\mathbb{R}}^{U(1)} \ , \ (V, r) \mapsto (\mathcal{E}_{\operatorname{gr}}^{U(1)})^{-1}(\bigoplus_{i \in \mathbb{Z}} (\operatorname{gr}^{U(1)})^i (\operatorname{HN}^{U(1)} \circ \omega(V, r)).$$

And then $\alpha^{U(1)}$ gives an isomorphism

$$\omega \cong \mathcal{E}^{U(1)} \circ \omega'.$$

□

Chapter 7

Shimura data in terms of $B(G,\mathbb{R})$

Let G be a reductive group over $\mathbb{R}$. For a homomorphism

$$h : \mathbb{S} \to G$$

write $X_h = \{ghg^{-1} \mid g \in G(\mathbb{R})\}$ for the set of its $G(\mathbb{R})$ conjugacy class. Furthermore we denote by $\mathbb{X}(G_\mathbb{C})$ the set of $G(\mathbb{C})$-conjugacy classes of cocharacters $\mu : \mathbb{G}_m \to G_\mathbb{C}$.

Recall the first axiom

- SV1: The Hodge structure on $\mathrm{Lie}(G_\mathbb{C})$ induced by $\mathrm{Ad} \circ h$ is of type $(1,-1),(0,0),(-1,1)$.

The aim of this chapter is to get a map

$$\big\{X_h \mid h : \mathbb{S} \to G \text{ fulfilling SV1}\big\} \longrightarrow \big\{(b,[\mu]) \in B(G,\mathbb{R})_{bsc} \times \mathbb{X}(G_\mathbb{C}) \mid \mu \text{ minuscule, } \kappa_G(b) = [\mu]\big\}$$

and show how to refine it to possibly get a bijection.

7.1 Definition of the map

Construction 7.1.1. There is a natural choice for an element

$$\gamma \in B(\mathbb{S},\mathbb{R}),$$

which will be motivated later on. First recall that there is an isomorphism

$$\mathbb{S}_\mathbb{C} \cong \mathbb{G}_m \times \mathbb{G}_m.$$

This isomorphism endows $\mathbb{G}_m \times \mathbb{G}_m$ with the involution

$$\begin{aligned} \sigma_\mathbb{S} : \mathbb{G}_m \times \mathbb{G}_m &\to \mathbb{G}_m \times \mathbb{G}_m \\ (z_1,z_2) &\mapsto (\bar{z}_2,\bar{z}_1). \end{aligned}$$

Then consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m(\mathbb{C}) & \longrightarrow & W & \longrightarrow & \Gamma & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle id\times id} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\| & & \\ 0 & \longrightarrow & \mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C}) & \longrightarrow & (\mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})) \rtimes \Gamma & \longrightarrow & \Gamma & \longrightarrow & 0 \end{array}$$

where $\alpha(j) = (1, -1)$. It is not difficult to check that the above diagram commutes. The class of this diagram is $[\gamma] \in B(\mathbb{S}, \mathbb{R})$. It is straight forward that γ is basic, since $\mathbb{G}_m \times \mathbb{G}_m$ is abelian.

Theorem 7.1.2. *The assignment*

$$\{X_h \mid h : \mathbb{S} \to G \text{ fulfilling SV1}\} \longrightarrow \{(b, [\mu]) \in B(G, \mathbb{R})_{bsc} \times \mathbb{X}(G_{\mathbb{C}}) \mid \mu \text{ minuscule, } \kappa_G(b) = [\mu]\}$$

which sends X_h to $\mu = h_{\mathbb{C}}(z, 1)$ and $[b] = [h_*(\gamma)]$ is well-defined.

We break up the proof of this theorem into several pieces. Also we set $\mu = h_{\mathbb{C}}(z, 1)$ and $[b] = [h_*(\gamma)]$ for the rest of the section.

Lemma 7.1.3. *In the above notation μ is minuscule, i.e. only the weights $-1, 0, 1$ occur in the representation*

$$\mathbb{G}_m \xrightarrow{\mu} G_{\mathbb{C}} \xrightarrow{\mathrm{Ad}_{\mathbb{C}}} \mathrm{GL}(\mathrm{Lie}(G)_{\mathbb{C}})$$

Proof. A representation

$$h : \mathbb{S} \to \mathrm{GL}(V)$$

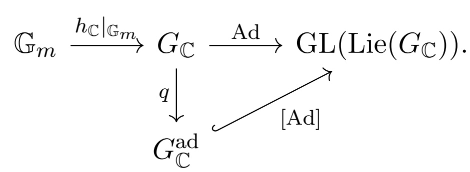
corresponds to the decomposition $V_{\mathbb{C}} = \bigoplus_{(p,q) \in \mathbb{Z}^2} V^{p,q}$, such that

$$h_{\mathbb{C}}(z_1, z_2)v = z_1^{-p} z_2^{-q} \text{ for } v \in V^{p,q}.$$

Thus given a decomposition $V = V^{1,-1} \oplus V^{0,0} \oplus V^{-1,1}$ implies that for the representation of $h_{\mathbb{C}}(z, 1)$ only the weights $1, 0, -1$ occur.
The claim in the lemma follows then from the special case $V = \mathrm{Lie}(G)$. □

Lemma 7.1.4. *In the above notation $[b]$ is basic.*

Proof. Consider the following commutative diagram



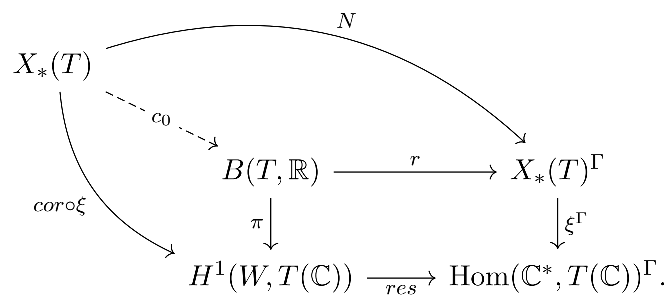
If we can show that the composition in the upper row is trivial, then, since $G_{\mathbb{C}}^{\mathrm{ad}} = G_{\mathbb{C}}/Z(G_{\mathbb{C}})$ embeds into $\mathrm{GL}(\mathrm{Lie}(G_{\mathbb{C}}))$, $q \circ h_{\mathbb{C}} \mid_{\mathbb{G}_m}$ (cf. Theorem 5.2.3.) is trivial as well and thus $h_{\mathbb{C}} \mid_{\mathbb{G}_m}$ factors through $Z(G_{\mathbb{C}})$.

As in the proof of the Lemma above we can see that $\mathbb{G}_m$ acts on the subspaces $\mathrm{Lie}(G_{\mathbb{C}})^{p,q}$ by multiplication by z^{q-p}. But by SV1 the only subspaces that occur are of weight $(1, -1), (0, 0), (-1, 1)$ and $\mathbb{G}_m$ acts trivially on all of them. Hence $\mathbb{G}_m$ acts trivially on $\mathrm{Lie}(G_{\mathbb{C}})$ and we are done. □

Lemma 7.1.5. *In the above notation we have $\kappa_G([b]) = [\mu]$.*

Proof. We prove this theorem in three steps analogous to Construction 3.3.9.

1. $G = T$ is a torus. In this case $\kappa_G = c^{-1}$. Thus we have to check that $c(\mu) = h_*(\gamma)$ inside $B(T, \mathbb{R})$. We recall the following construction from Section 3.3.:



Since c is defined via c_0, it suffices to show that $c_0(\mu) = h_*(\gamma)$. We first compute the norm of μ

$$\begin{aligned} N(\mu) = \mu \cdot \sigma.\mu = h_{\mathbb{C}}(z,1) \cdot \sigma_T(h_{\mathbb{C}}(\sigma(z),1)) &= h_{\mathbb{C}}(z,1) \cdot h_{\mathbb{C}}(\sigma_{\mathbb{S}}(\sigma(z),1)) \\ &= h_{\mathbb{C}}(z,1) \cdot h_{\mathbb{C}}(1,z) \\ &= h_{\mathbb{C}}(z,z). \end{aligned}$$

For the second map, one can check from the definition of ξ (cf. discussion after Definition 3.2.7.) that

$$\begin{aligned} \xi : X_*(T) &\to \mathrm{Hom}(\mathbb{C}^*, T(\mathbb{C})) \\ v &\mapsto v(\mathbb{C}). \end{aligned}$$

Let us now compute the corestriction map in our case (or rather its evaluation at $j \in W$) as stated in Definition 3.2.4. We choose the section $\Gamma \to W$ which sends the non-trivial element to j.

$$\mathrm{cor}(\mu(\mathbb{C}))(j) = \sigma(\mu(j^{-1}j)) \cdot \mu(j^2) = h(1,1) \cdot h(-1,1) = h(-1,1).$$

Now a comparison with the definition of $h_*(\gamma)$ shows that $c_0(\mu)$ defines the same element inside $B(T, \mathbb{R})$. To be precise, we have an equality

$$h(1,-1).c_0(\mu) = h_*(\gamma)$$

inside $Z^1_{alg}(W, T(\mathbb{C}))$.

2. Now assume that the derived subgroup of G is simply connected. Write

$$q : G \to D$$

for the quotient of D by its derived subgroup. Then κ_G is defined via this diagram:

$$\begin{array}{ccc} B(G,\mathbb{R}) & \xrightarrow{\kappa_G} & \pi_1(G)_\Gamma \\ \downarrow & & \| \\ B(D,\mathbb{R}) & \xrightarrow{\kappa_D} & \pi_1(D)_\Gamma \end{array}$$

This means it suffices to prove that $\kappa_D(q_* h_*(\gamma)) = [q \circ \mu]$, which has been shown in the first case.

3. In the general case use a z-extension

$$1 \to Z \to G' \xrightarrow{p} G \to 1$$

and in this case κ_G is defined via this diagram.

$$\begin{array}{ccc} B(G',\mathbb{R}) & \xrightarrow{\kappa_{G'}} & \pi_1(G')_\Gamma \\ \downarrow p & & \downarrow p \\ B(G,\mathbb{R}) & \xrightarrow{\kappa_G} & \pi_1(G)_\Gamma \end{array} \tag{7.1}$$

Now we want to choose a lift μ' of μ. It can be constructed as follows. Choose a maximal torus $\mathrm{im}(\mu) \subset T \subset G$. Since $G' \to G$ is surjective, by [Hum12, Section 21.3.] we can choose a maximal torus $T' \subset G'$, such that $p(T') = T$. In this case we have a surjection

$$X_*(T') \xrightarrow{p_*} X_*(T)$$

and this allows us to choose a lift of μ. Having chosen a lift, let us check that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & W & \longrightarrow & \Gamma & \longrightarrow & 0 \\ & & \downarrow (\mu'\cdot\sigma.\mu') & & \downarrow \alpha & & \| & & \\ 0 & \longrightarrow & G'(\mathbb{C}) & \longrightarrow & G'(\mathbb{C}) \rtimes \Gamma & \longrightarrow & \Gamma & \longrightarrow & 0, \end{array}$$

where $\alpha(j) = \sigma.\mu'(-1)$, is commutative. Using that

$$\mathrm{im}(\mu' \cdot \sigma.\mu') \subset p^{-1}(\mathrm{im}(\mu \cdot \sigma.\mu))$$

we have that by [Kot14, Lemma 10.5.] $(\mu' \cdot \sigma.\mu')$ factors through a central torus. Thus commutativity of the diagram is equivalent to verifiying that $(\mu' \cdot \sigma.\mu')$ is equal to $\sigma.(\mu' \cdot \sigma.\mu') = (\sigma.\mu' \cdot \mu')$. But since the cocharacter is central we have

$$(\mu' \cdot \sigma.\mu')\mu'^{-1} = \mu'^{-1}(\mu' \cdot \sigma.\mu') = \sigma.\mu'$$

which implies the desired result. We call the equivalence class of this element $b' \in B(G',\mathbb{R})$. We claim that

- $p_*(b') = b$
- $\kappa_{G'}(b') = [\mu']$.

The first claim follows since by construction

$$p \circ \mu' = \mu.$$

As for the second claim, we would like to apply the second case treated before. This is not exactly possible, since μ' is not necessarily of the form $h'_{\mathbb{C}}(z,1)$. But we can use a very similar calculation made in the first step to see that the equality holds assuming that G' is a torus and then apply the conclusion as in the second case when the derived subgroup of G' is simply connected. Since by construction this is always the case, we conclude that

$$\kappa_{G'}(b') = [\mu'].$$

To summarize we have by the commutative diagram (7.1)

$$\kappa_G(h_*(\gamma)) = \kappa_G(p_*(b')) = p_*(\kappa_{G'}(b')) = p_*([\mu']) = [\mu].$$

This concludes the proof.

□

7.2 Flag varieties and modifications

Now we want to construct a map

$$\big\{(b,[\mu]) \in B(G,\mathbb{R})_{bsc} \times \mathbb{X}(G_{\mathbb{C}}) \mid \mu \text{ minuscule, } \kappa_G(b)=[\mu]\big\} \to \big\{K \subset \mathrm{Hom}^{\otimes}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}))\big\}$$

using the results from the previous chapters.

Construction 7.2.1. Start with such a couple $(b,[\mu])$. In general to a cocharacter μ we can associate a parabolic subgroup

$$P_\mu = \{g \in G \mid \lim_{t\to 0} \mathrm{ad}(\mu(t))g \text{ exists}\}.$$

The flag variety associated to μ is defined as

$$\mathrm{Fl}_{G,\mu} = G/P_\mu.$$

We want to define a map

$$\mathrm{Fl}_{G,\mu}(\mathbb{C}) = G(\mathbb{C})/P_\mu(\mathbb{C}) \to \{U(1)\text{-equivariant } G\text{-bundles}\}.$$

Let us first consider the $G = GL(V_{\mathbb{C}})$ case, where V is real vector space of dimension m. In this case we can pick μ to be of the form

$$z \mapsto \begin{pmatrix} z^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & z^{\lambda_n} \end{pmatrix}$$

with

$$\lambda_1 > ... > \lambda_n.$$

Furthermore we set $a_k := dim(V_{\mathbb{C},k})$, the dimension of the subspace on which the $\mathbb{G}_m$ acts through the character z^{λ_k}. The flag variety $\mathrm{Fl}_{Gl(V_{\mathbb{C}}),\mu}(\mathbb{C})$ parameterizes all flags of the form

$$0 = W_0 \subset W_1 \subset ... \subset W_n = V_{\mathbb{C}}\ ,\ dim(W_{k+1}/W_k) = a_k.$$

To such a flag we can associate the following lattice

$$\sum_{k\in\mathbb{Z}} t^{-\lambda_k} W_k[[t]] \subset V_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}(\!(t)\!).$$

Such lattices correspond to $U(1)$-equivariant vector bundles. Thus we get a map

$$\begin{aligned} d_{G,\mu} : \mathrm{Fl}_{G,\mu}(\mathbb{C}) &\to \{U(1)\text{-equivariant vector bundles}\} \\ x &\mapsto \mathcal{E}_x \end{aligned}$$

Note that while different choices of μ can give the same flag variety, $d_{G,\mu}$ depends strictly on μ.

We can generalise this description for a general reductive group G. In this case giving an element in $x \in \mathrm{Fl}_{G,\mu}(\mathbb{C})$ is equivalent to giving for each representation $(V,r) \in \mathrm{Rep}_{\mathbb{R}}(G)$ a flag

$$0 \subset W_0 \subset ... \subset W_r = V_{\mathbb{C}}$$

conjugate to the flag induced by $\mathrm{Fil}^*(r \circ \mu)$. In the same spirit we can define

$$\begin{aligned} d_{G,\mu} : \mathrm{Fl}_{G,\mu}(\mathbb{C}) &\to \{U(1)\text{-equivariant } G\text{-bundles}\} \\ x &\mapsto \mathcal{E}_x : (V, r) \mapsto d_{GL(V), r\circ\mu}(V), \end{aligned}$$

where we regard $U(1)$-equivariant G-bundles as functors

$$\mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Bun}_X^{U(1)}.$$

And furthermore we define

$$Z_{b,\mu} := \{\mathcal{E}_x \mid x \in \mathrm{Fl}_{G,\mu}(\mathbb{C}) \text{ such that } \mathcal{E}_x \cong \mathcal{E}_b \text{ as } G\text{-bundles}\}$$

This allows us to define the map

$$\begin{aligned} \big\{(b,[\mu]) \in B(G,\mathbb{R})_{bsc} \times \mathbb{X}(G_{\mathbb{C}}) \mid \mu \text{ minuscule}, \kappa_G(b) = [\mu]\big\} &\to \big\{K \subset \mathrm{Hom}^{(}\,\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}))\big\} \\ (b,\mu) &\mapsto Z_{b,\mu} \end{aligned}$$

where we used Theorem 6.3.9. to identify semistable bundles with $\mathrm{Hom}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Rep}_{\mathbb{R}}(\mathbb{S})$.

Lemma 7.2.2. *The following diagram*

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}) & & \\ \downarrow{\scriptstyle \mathcal{H}} & \searrow^{\mathcal{R}} & \\ \mathrm{Isoc}_{\mathbb{R}}^{U(1)} & \xrightarrow{\mathcal{E}^{U(1)}} & \mathrm{Bun}_X^{U(1)} \\ {\scriptstyle Forget}\downarrow & & \downarrow{\scriptstyle Forget} \\ \mathrm{Isoc}_{\mathbb{R}} & \xrightarrow{\mathcal{E}} & \mathrm{Bun}_X \end{array}$$

commutes up to isomorphism.

Proof. The above triangle was shown to be commutative (up to isomorphism) in Proposition 6.2.10. The square commutes by construction of $\mathcal{E}^{U(1)}$ Lemma 6.2.7. □

Lemma 7.2.3. *Let $h : \mathbb{S} \to G$ be a morphism, $b = h_*(\gamma) \in B(G,\mathbb{R})$, and denote by*

$$i : X_h \to \mathrm{Fl}_{G,\mu}(\mathbb{C})$$

the Borel embedding. For $\mathcal{E}_{i(h)} = d_{G,\mu}(i(h))$ we have an isomorphism

$$\mathcal{E}_b \cong \mathcal{E}_{i(h)}$$

as vector bundles.

Proof. For the right hand side note that $\mathcal{R}$ was exactly defined to be

$$\mathcal{R} \circ h^* = \mathcal{E}_{i(h)}.$$

For the other side note that

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{R}}(G) & & \\ \downarrow{\scriptstyle h^*} & \searrow^{\mathcal{N}_b} & \\ \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}) & \xrightarrow[Forget\circ\mathcal{H}]{} & \mathrm{Isoc}_{\mathbb{R}} \end{array}$$

commutes up to isomorphism. Thus by Lemma 7.2.2.

$$\mathcal{E}_b = \mathcal{E} \circ N_b \cong \mathcal{E} \circ Forget \circ \mathcal{H} \circ h^*. \cong Forget \circ \mathcal{R} \circ h^* = \mathcal{E}_{i(h)}.$$

□

Corollary 7.2.4. *For* $(b, \mu) = (h_*(\gamma), h_{\mathbb{C}}(z, 1))$ *we have*

$$X_h \subset Z_{b,\mu}.$$

Proof. Since for $g \in G(\mathbb{R})$

$$\mathcal{E}_{i(h)} \cong \mathcal{E}_{i(g.h)}$$

we are done. □

7.3 The second axiom and the inner form J_b

In this section we want to collect ideas that can not be made more precise to give a rigorous proof. The first section is related to the SV2 axiom, which hasn't been involved yet. There is an evident connection to the inner form J_b which we will introduce and examine.

Before introducing the inner form J_b, we will recall a more classical construction.

Definition 7.3.1. Let G be a reductive group over $\mathbb{R}$. An involution over $\mathbb{R}$

$$\theta : G \to G$$

is said to be Cartan, if

$$G^{\theta}(\mathbb{C}) = \{g \in G(\mathbb{C}) \mid g = \theta(\bar{g})\}$$

is compact.

To each such involution we can associate a real algebraic group H, such that

1. $H_{\mathbb{C}} \cong G_{\mathbb{C}}$
2. $H(\mathbb{R}) = G^{\theta}(\mathbb{C})$.

This is achieved by taking $\theta \circ \sigma$ as descent datum for the complex reductive group $G_{\mathbb{C}}$. We will denote H by G^{θ}. Let us recall the second axiom SV2:

- $\mathrm{ad}(h(i))$ is a Cartan involution of G^{ad}.

We will now draw connection between this classical construction and an algebraic group defined in terms of $b \in B(G, \mathbb{R})$. We refer to [Kot14, 2.6.] for more details.

Construction 7.3.2. Let $b = (b_1, b_2) \in Z^1_{alg}(W, G(\mathbb{C})$ and denote by $G_b \subset G_{\mathbb{C}}$ the centralizer of $b_1 : \mathbb{G}_m \to G_{\mathbb{C}}$. Choose a lift $w \in W$ of $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then we can regard $x_w := b_2(w)$ as in element in $G_{\mathbb{C}}(\mathbb{C})$ and

$$\mathrm{ad}(x_w) : G_{\sigma(b)} \to G_b$$

is an isomorphism independent of the choice of the lifting w. Furthermore these isomorphisms are a descent datum for G_b. The descent of G_b is denoted by J_b. It is a real algebraic group over $\mathbb{R}$, such that

1. $J_b(\mathbb{C}) = G_{\mathbb{C}}(\mathbb{C}) = G(\mathbb{C})$
2. $J_b(\mathbb{R}) = \{g \in G(\mathbb{C}) \mid g = \sigma_{J_b}(g) = x_w \sigma(g) x_w^{-1}\}$.

We will be only interested in the case that b is basic. In this case J_b is a real form of $G_{\mathbb{C}}$.

Proposition 7.3.3. If $b' = g.b$, then $J_b \cong J_{b'}$.

Proof. One checks that the following diagram

$$\begin{array}{ccc} G_{\mathbb{C}} & \xrightarrow{\mathrm{ad}(g)} & G_{\mathbb{C}} \\ \sigma_{J_b}\downarrow & & \downarrow\sigma_{J_{g.b}} \\ G_{\mathbb{C}} & \xrightarrow{\mathrm{ad}(g)} & G_{\mathbb{C}} \end{array}$$

commutes. In particular the isomorphism $\mathrm{ad}(g)$ descends to an isomorphism $J_b \cong J_{g.b}$ over $\mathbb{R}$. □

Now assume we have a morphism

$$h : \mathbb{S} \to G.$$

Recall that we defined $b = h_*(\gamma)$ and as we have seen that b is basic. Thus we can apply the above construction to get a real form J_b of $G_{\mathbb{C}}$. We will make this description more concrete. We can choose $w := i \cdot j \in W_{\mathbb{C}/\mathbb{R}}$ as preimage of $\sigma \in \Gamma$. In this case

$$x_w = h_{\mathbb{C}}(i, i) \cdot h_{\mathbb{C}}(1, -1) = h_{\mathbb{C}}(i, -i) = h(i)$$

since

$$\begin{aligned} \mathbb{S}(\mathbb{R}) \cong \mathbb{C}^* &\to \mathbb{S}(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}^* \\ z &\mapsto (z, \bar{z}). \end{aligned}$$

Thus J_b is the real form of $G_{\mathbb{C}}$ with $\sigma_{J_b} = \mathrm{ad}(h(i)) \circ \sigma$. Thus

$$J_b \cong G^{\theta}$$

and SV2 says that $J_b^{\mathrm{ad}}(\mathbb{R})$ is compact.

Let us point one small subtlety here. To be able to conclude as above we need to identify $G^{\theta}/Z(G^{\theta}) \cong (G/Z(G))^{\theta}$, which can seen easily by writing out an isomorphism over $\mathbb{C}$ which respects the descent data.

We give one more description of J_b;

Lemma 7.3.4. *Consider the fiber functor*

$$\mathcal{E}_b : \mathrm{Rep}_{\mathbb{R}}(G) \xrightarrow{N_b} \mathrm{Isoc}_{\mathbb{R}} \xrightarrow{\mathcal{E}} \mathrm{Bun}_X .$$

Then there is an isomorphism

$$\underline{\mathrm{Aut}}(\mathcal{E}_b) \cong J_b$$

as sheaves of groups over $(Sch/X)_{\acute{e}t}$.

Proof. We can identify

$$\mathrm{Bun}_X \cong \mathrm{Bun}(\pi : \mathbb{P}^1_{\mathbb{C}} \to X).$$

We claim that the automorphisms of the functor

$$\mathrm{Rep}_{\mathbb{R}}(G) \xrightarrow{\omega_b} \mathrm{Bun}_X \xrightarrow{\pi^*} \mathrm{Bun}_{\mathbb{P}^1_{\mathbb{C}}}$$

are isomorphic to $G(\mathbb{P}^1_{\mathbb{C}})$: Since b is basic, $\mathrm{Rep}_{\mathbb{R}}(G)$ is endowed with a $\mathbb{Z}$ grading. Thus any automorphism

$$\alpha : \omega_b \otimes \mathbb{C} \to \omega_b \otimes \mathbb{C}$$

splits as direct sum $\oplus \alpha_n$. The same holds in fact for any automorphism

$$\alpha' : \omega^G \otimes \mathbb{C} \to \omega^G \otimes \mathbb{C}.$$

Since

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n), \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)) \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}),$$

we have an isomorphism

$$\underline{\mathrm{Aut}}(\omega_b \otimes \mathbb{C}) \to \underline{\mathrm{Aut}}(\omega^G \otimes \mathbb{C}).$$

As the latter is isomorphic to $G(\mathbb{P}^1_{\mathbb{C}})$ by standard Tannakian formalism, the claim follows. If we set $b_2(j) = (A, \sigma)$, we see that the descent condition

$$\begin{array}{ccc} f^*((\omega_b \otimes \mathbb{C})(V,r)) & \xrightarrow{f^*(r_{\mathbb{C}}(g))} & f^*((\omega_b \otimes \mathbb{C})(V,r)) \\ {\scriptstyle (r_{\mathbb{C}}(A)\circ\sigma)\otimes c_1}\downarrow & & \downarrow{\scriptstyle (r_{\mathbb{C}}(A)\circ\sigma)\otimes c_1} \\ (\omega_b \otimes \mathbb{C})(V,r) & \xrightarrow{r_{\mathbb{C}}(g)} & (\omega_b \otimes \mathbb{C})(V,r) \end{array}$$

translates to

$$r_{\mathbb{C}}(gA) = r_{\mathbb{C}}(A\bar{g})$$

for all representations and thus it translates to

$$gA = A\bar{g},$$

which is exactly the condition that $g \in J_b(X)$.

Given an étale morphism $i : U \to X$ we can rerun the whole argument with

$$\mathrm{Bun}_U \cong \mathrm{Bun}(\pi_U : U_{\mathbb{C}} \to U)$$

to see that an automorphism is given by $g \in G(U_{\mathbb{C}})$ which commutes with A, i.e. $g \in J_b(U)$. □

7.4 Prospects

The aim of this section is to sketch how to use the prior results in this thesis to improve the results. We begin with the following Conjecture;

Conjecture 2. *The map*

$$\{X_h \mid h : \mathbb{S} \to G \text{ fulfilling SV1+SV2}\} \to \{(b, [\mu]) \in B(G, \mathbb{R})_{bsc} \times \mathbb{X}(G_{\mathbb{C}}) \mid \mu \text{ minuscule, } \kappa_G(b) = [\mu]\}$$

is injective.

It is sufficient to prove that the inclusion in Corollary 7.2.4.

$$X_h \subset Z_{b,\mu}$$

is an equality. By Lemma 7.3.4. we have an isomorphism $\underline{Aut}(\mathcal{E}_b) \cong J_b$ and thus a bijection

$$\begin{aligned} H^1_{\acute{e}t}(X,G) &\xrightarrow{\sim} H^1_{\acute{e}t}(X,J_b) \\ \mathcal{P} &\mapsto Isom(\mathcal{E}_b,\mathcal{P}). \end{aligned}$$

Under this bijection $\mathcal{E}_x$ (cf. Construction 7.2.1) is sent to $\mathcal{E}_{b,x}$.

The idea now is to consider the latter modifications as modifications via

$$\mathrm{Fl}_{J_b,\mu^{-1}}(\mathbb{C}).$$

Since SV2 implies that J_b^{ad} is compact, we can apply the following Lemma. We denote by the superscript $^+$ the connected component in the euclidean topology;

Lemma 7.4.1. *Let G be a reductive group over $\mathbb{R}$ such that G^{ad} is compact. Then the action*

$$G(\mathbb{R})^+ \curvearrowright G_{\mathbb{C}}(\mathbb{C})/P_\mu(\mathbb{C})$$

is transitive.

Proof. We can reduce this lemma to proving that

$$G^{\mathrm{ad}}(\mathbb{R})^+ \curvearrowright G^{\mathrm{ad}}_{\mathbb{C}}(\mathbb{C})/P^{\mathrm{ad}}_\mu(\mathbb{C})$$

is transitive, since the latter is isomorphic to $G_{\mathbb{C}}(\mathbb{C})/P_\mu(\mathbb{C})$ and the former admits a surjection $G(\mathbb{R})^+ \twoheadrightarrow G^{\mathrm{ad}}(\mathbb{R})^+$. In this case we can apply a slight modification of [Wol69, Theorem 2.6.(3)], since $G^{\mathrm{ad}}(\mathbb{R})^+$ is a finite index subgroup of $G^{\mathrm{ad}}(\mathbb{R})$, to conclude that there is an open orbit $G^{\mathrm{ad}}(\mathbb{R})^+.x$, which is necessarily closed since $G^{\mathrm{ad}}(\mathbb{R})^+$ is compact. Thus

$$G^{\mathrm{ad}}(\mathbb{R})^+.x = G^{\mathrm{ad}}_{\mathbb{C}}(\mathbb{C})/P^{\mathrm{ad}}_\mu(\mathbb{C}).$$

□

Now if $x_1, x_2 \in \mathrm{Fl}_{G,\mu}(\mathbb{C})$ such that $\mathcal{E}_{x_1} \cong \mathcal{E}_{x_2}$, then $\mathcal{E}_{x_1,x_1} \cong \mathcal{E}_{x_1,x_2}$ as J_b torsors. By the above Lemma, we should be able to find an element in $g \in J_b(\mathbb{R})$, since both are modifications via $\mathrm{Fl}_{J_b,\mu^{-1}}(\mathbb{C})$, such that the isomorphism $\mathcal{E}_{x_1,x_1} \cong \mathcal{E}_{x_1,x_2}$ is given by g_*. This should imply that we can find an isomorphism $\mathcal{E}_{x_1} \xrightarrow{g'_*} \mathcal{E}_{x_2}$ given by $g' \in G(\mathbb{R})$ and thus

$$Z_{b,\mu} \subset X_h.$$

Finally let us remark that to get a bijection

$$\big\{X_h \mid h: \mathbb{S} \to G \text{ fulfilling SV1+SV2}\big\} \longrightarrow \big\{(b,[\mu]) \in B(G,\mathbb{R})_{bsc} \times \mathbb{X}(G_{\mathbb{C}}) \mid \mu \text{ minuscule}, \kappa_G(b) = [\mu]\big\} \\ + \text{ further conditions}$$

we did define a possible inverse in Section 7.2.

$$\begin{aligned} \big\{(b,[\mu]) \in B(G,\mathbb{R})_{bsc} \times \mathbb{X}(G_{\mathbb{C}}) \mid \mu \text{ minuscule}, \kappa_G(b) = [\mu]\big\} &\to \big\{K \subset \mathrm{Hom}^{\otimes}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}))\big\} \\ (b,\mu) &\mapsto Z_{b,\mu}. \end{aligned}$$

From the first part of this section we see, that necessarily we need to include a condition that corresponds to SV2. One such condition might be

$$| B(J_b, \mu^{-1}) |= 1$$

relating modifications of bundles bounded by a character μ to a more intrinsic description inside $B(G, \mathbb{R})$. It is not clear how to exactly define $B(J_b, \mu^{-1})$.

Furthermore the functors

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_{\mathbb{R}}(G), \mathrm{Rep}_{\mathbb{R}}(\mathbb{S}))$$

a priori only yield morphisms

$$h : \mathbb{S} \to H$$

where H is an inner form of G. Thus $\kappa_G(b) = [\mu]$ might be necessary to conclude that indeed the inner form in question is G.

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