STRATAS OF Bun_G

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ABSTRACT. These are notes of a talk that I gave in the London number theory study group on the Geometrization of the local Langlands correspondence. The first three sections cover classical notions around Kottwitz' set $B(G)$, standard references being [RR96], [Kot85]& [Kot97]. The rest follows Chapter III of [FS21].

CONTENTS

1. ISOCRYSTALS AND VECTOR BUNDLES

One of the initial instances where isocrystals were observed, was through crystalline cohomology and the classification of p-divisible groups over perfect fields in positive characteristic.

For this let $k = \overline{\mathbb{F}}_p$. Then we may take $W(k)$, the Witt vector ring of k, and $K := W(k)[\frac{1}{p}]$. In fact one can check that $K = \breve{\mathbb{Q}}_p$, the completion of the maximal unramified extension of $\mathbb{Q}_p$. By abuse of notation let us denote by σ both the Frobenius of k and its lift to $W(k)$ and K.

Definition 1.1. (1) An isocrystal is a K vector space V with an σ-semilinear isomorphism, i.e. an isomorphism

$$\phi : V \to V$$

such that for $\lambda \in K$, $v \in V$

$$\phi(\lambda v) = \sigma(\lambda)\phi(v).$$

(2) The category of isocrystals, denoted by $\mathrm{Isoc}_{\mathbb{Q}_p}$, has as objects isocrystals and morphisms are morphisms of K-vector spaces that commute with the semilinear isomorphisms.

The following is a result of Dieudonné and Manin

Theorem 1.2. (1) *There is a natural isomorphism*

$$\{p\text{-divisible group over } k\}/\text{up to isogeny} \to \{\text{isocrystals}\}/\text{up to isomorphism}$$

(2) *The category* $\mathrm{Isoc}_{\mathbb{Q}_p}$ *is semisimple and the simple factors are indexed by* $\mathbb{Q}$*:*

$$E^{\frac{m}{n}} = (K^n, \begin{bmatrix} 0 & 0 & \dots & p^m \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 \\ 0 & \dots & 1 & 0 \end{bmatrix} \circ \sigma)$$

where m *and* n *are coprime integers and* $n > 0$.

As a corollary one can see that

$$\mathrm{Isoc}_{\mathbb{Q}_p} = \bigoplus_{\lambda \in \mathbb{Q}} \mathrm{Isoc}_{\mathbb{Q}_p}^{\lambda}$$

where $\mathrm{Isoc}_{\mathbb{Q}_p}^{\lambda}$ is the subcategory spanned by E^{λ}. Furthermore, one can check that $\mathrm{End}_{\mathrm{Isoc}_{\mathbb{Q}_p}}(E^{\lambda}) = D_{\lambda}$, this is the division algebra corresponding to $\overline{\lambda} \in \mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$.

Now we can approach the first definition of $B(G)$. Consider the map

$$\mathrm{GL}_n(K) \to \{\text{isocrystals of } \dim = n\}/\cong$$
$$b \mapsto (K^n, b \circ \sigma)$$

Essentialy by the above theorem this map is surjective, but it is not injective. One can check that b and b' map to isomorphic isocrystals if and only if $b = gb'\sigma(g)^{-1}$ for some $g \in \mathrm{GL}_n(K)$, this is also called σ-equivalence.

Definition 1.3. Define $B(\mathrm{GL}_n) := \mathrm{GL}_n(K)/\sim$, where $\sim$ refers to the σ-equivalence.

Remark 1.4. Thus $B(\mathrm{GL}_n) \cong \{\text{isocrystals of } \dim = n\}/\cong$.

Let us point out a first geometric interpretation of the above discussion, that will be relevant to us. Let $S \in \mathrm{Perf}_{\mathrm{k}}$ and $X_S = X_{S,\mathbb{Q}_p}$ the Fargues-Fontaine curve, which can be written as a quotient $X_S = Y_S/\varphi_S^{\mathbb{Z}}$. Then we get a functor

$$\mathcal{E} : \mathrm{Isoc}_{\mathbb{Q}_p} \to \mathrm{Bun(X_S)} \quad (1.1)$$
$$(D, \phi) \mapsto (D \otimes_K \mathcal{O}_{Y_S}, \phi \otimes \varphi_S).$$

A priori $\mathcal{E}(D, \phi)$ is a (trivial) vector bundle over Y_S with a descent datum, thus using descent we may regard it as vector bundles on X_S.

Theorem 1.5. *Let C be a complete algebraically closed non-archimedean field over k. Then the above functor*

$$\mathcal{E} : \mathrm{Isoc}_{\mathbb{Q}_p} \to \mathrm{Bun(X_C)}$$

is essentially surjective and faithful.
It is fully faithful, when restricted to $\mathrm{Isoc}_{\mathbb{Q}_p}^{\lambda}$.

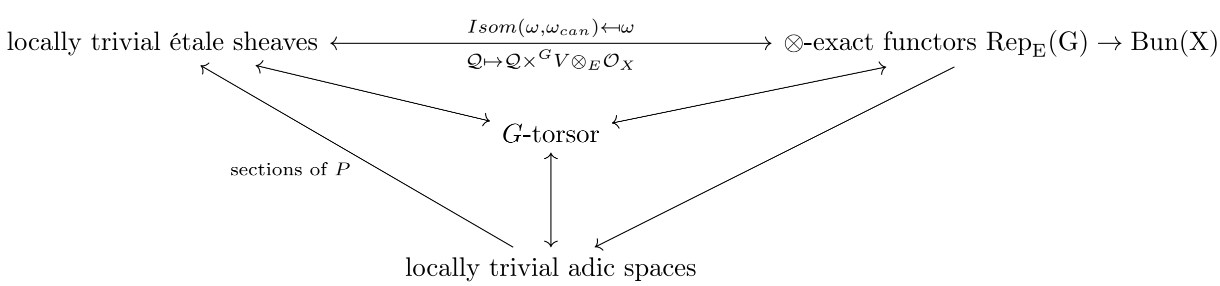
Remark 1.6. The failure of the functor to be full in general is given by non trivial homomorphisms of different slopes in the category of vector bundles, e.g. $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))$.

2. G-isocrystals, G-bundles and $B(G)$

We will now use the language of Tannakian formalism to generalize the prior results. Let us also fix for the rest of these notes

(1) E, an nonarchimedean local field with its Frobenius $\sigma = \sigma_E$ (this was $\mathbb{Q}_p$ in the prior section)
(2) Γ, this is the Galois group $\mathrm{Gal}(\breve{E}/E)$
(3) G, a reductive group over E
(4) k, a complete algebraically closed non-archimedean field over $\mathbb{F}_q$

As we will work with G-torsors the whole time, let us briefly recap what they are. We find three equivalent definitions of a G-torsor over a sousperfectoid spaces over E, although there are other situations where such equivalences apply. For a complete account of this material see [SW20, Theorem 19.5.2].



For us the "canonical" way to think of G-torsors will be as ⊗-exact functors or as locally trivial étale sheaves. Sometimes also the term G-bundle is used.

With this in mind, let us define G-isocrystals. Here we extend our notion of $\mathrm{Isoc}_{\mathbb{Q}_p}$ from the last section to E (by looking at vector spaces over $\breve{E}$ and imposing the σ-semilinearity condition). So from now on we write $\mathrm{Isoc} = \mathrm{Isoc}_E$ for the corresponding category.

Definition 2.1. A G-isocrystal is a $\otimes$-exact tensor functor

$$\mathrm{Rep_E(G)} \to \mathrm{Isoc}.$$

Now we may *define* the set $B(G)$ as isomorphism classes of such G-isocrystals. Equivalently, we might naturally extend the definition made in 1.3:

Definition 2.2. Define

$$B(G) := G(\breve{E})/\sim$$

where $b, b' \in G(\breve{E})$ are equivalent, if there is a $g \in G(\breve{E})$ such that

$$b = gb'\sigma(g)^{-1}.$$

Remark 2.3. To be precise, one has to write $B(G, E)$, but as it is usually clear from the context, the field E is omitted.

Lemma 2.4. *The map*

$$B(G) \to \{\otimes\text{-}exact\ functors\ \mathrm{Rep_E(G)} \to \mathrm{Isoc}\}/\cong$$
$$b \mapsto N_b : (V, r) \mapsto (V \otimes_E \breve{E}, b(id \otimes \sigma))$$

is an isomorphism

Remark 2.5. The more general and careful way to state this lemma and the prior definition is to require that the $\otimes$-exact tensor functors ω after composing with the forgetful functor $\mathrm{Isoc} \to \mathrm{Vect}_{\breve{\mathrm{E}}}$ are isomorphic to the trivial fibre functor $\omega^G \otimes_E \breve{E}$. But Steinberg's theorem ensures that in our case $H^1(\breve{E}, G) = *$ and thus every fibre functor $\mathrm{Rep_E(G)} \to \mathrm{Vect}_{\breve{\mathrm{E}}}$ is trivial.

Now it is also possible to generalize Theorem 1.5

Theorem 2.6. *Let C be a complete algebraically closed non-archimedean field over k. Postcomposing with the functor $\mathcal{E}$ (from the above theorem) yields a functor*

$$\{\otimes\text{-}exact\ functors\ \mathrm{Rep_E(G)} \to \mathrm{Isoc}\} \xrightarrow{\circ\mathcal{E}} \{\otimes\text{-}exact\ functors\ \mathrm{Rep_E(G)} \to \mathrm{Bun(X_C)}\} = \mathrm{Bun_G(C)}$$

from G-isocrystals to G-torsors on X_C. It induces an isomorphism on the set of isomorphism classes of the respective categories.
In particular $B(G)$ classifies G-torsors on X_C (up to isomoprhism).

This allows one to conclude

Corollary 2.7. *There is a (functorial) bijection of sets $B(G) \cong |\mathrm{Bun_G}|$.*

Remark 2.8. It is possible to endow $B(G)$ with a topology induced from its partial ordering. It turns out to be that the above bijection is actually a homeomorphism (where $|\mathrm{Bun_G}|$ is the topological space coming from a stack), see [Vie21].

3. Properties and invariants of $B(G)$

Now let us study the set $B(G)$ closer. Two reasons why one might be interested in this:

(1) Let S be a scheme over $\mathbb{F}_p$ with a p-divisible group G, e.g. special fibres of certain Shimura varieties, then one can show that the map

$$S \to B(G)$$
$$s \mapsto G_{\bar{s}}$$

induces a stratification, i.e. $S = \coprod_{B(G)} S_b$ is a decomposition into locally closed subset.

(2) Let $S \in \mathrm{Perf_k}$ and $\mathcal{P} \in \mathrm{Bun_G(S)}$ a G-torsor. This induces a map

$$S \to |\mathrm{Bun_G}| \to \mathrm{B(G)}$$
$$s \mapsto \quad \mathcal{P}_{\bar{s}} \quad \mapsto \text{Iso. class of } \mathcal{P}_{\bar{s}}$$

This is the case we are interested in and which we will study closer.

There are two invariants of that are important for our study of $B(G)$, the Newton point and the Kottwitz point.

3.1. **The Newton point.** The first invariant that we want to define is a map

$$\nu : B(G) \to \mathrm{Hom}_{\breve{E}}(\mathbb{D}_{\breve{E}}, G_{\breve{E}})/G(\breve{E})\text{-conjugacy.} \tag{3.1}$$

Here $\mathbb{D}$ is the pro-algebraic torus, such that $X^*(\mathbb{D}) \cong \mathbb{Q}$. We can explicitly build it as

$$\mathbb{D} = \varprojlim_{n\in\mathbb{N}} \mathbb{G}_m \xleftarrow{\cdot 2} \mathbb{G}_m \xleftarrow{\cdot 3} \mathbb{G}_m \leftarrow ...$$

(which is dual to $\mathbb{Q} = \varinjlim_{n\in\mathbb{N}} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \to ...$). But the more convenient way to think of this is that we have an equivalence of Tannakian categories

$$\mathrm{Rep}(\mathbb{D}) \cong \{\mathbb{Q}\text{-graded vector spaces}\}$$

extending the equivalence

$$\mathrm{Rep}(\mathbb{G}_\mathrm{m}) \cong \{\mathbb{Z}\text{-graded vector spaces}\}.$$

Now if $G = \mathrm{GL}_n$, then we can easily define the map (3.1): An element $b \in B(\mathrm{GL}_n)$ corresponds to an isocrystal (V, ϕ) of dimension n. By the Dieudonné-Manin classifcation, this automatically comes with an slope decomposition $V = \bigoplus_{i=1}^{r} V^{\lambda_i}$, where $\lambda_i \in \mathbb{Q}$. But such a $\mathbb{Q}$-linear decomposition gives rise to an element in $\mathrm{Hom}_{\breve{E}}(\mathbb{D}_{\breve{E}}, G_{\breve{E}})/G(\breve{E})$-conjugacy. More generally we have a functor

$$\mathrm{slope} : \mathrm{Isoc} \to \{\mathbb{Q}\text{-graded vector spaces}\} \cong \mathrm{Rep}_{\breve{\mathrm{E}}}(\mathbb{D})$$

which sends an isocrystal to its slope decomposition. Postcomposition with this functor yields

$$\{\otimes\text{-exact functors } \mathrm{Rep}_\mathrm{E}(\mathrm{G}) \to \mathrm{Isoc}\} \xrightarrow{\circ\mathrm{slope}} \{\otimes\text{-exact functors } \mathrm{Rep}_\mathrm{E}(\mathrm{G}) \to \{\mathbb{Q}\text{-graded vector spaces}\}\}.$$

and taking isomorphism classes we get our desired map Equation (3.1). This can be refined to a map, which we also call ν,

$$\nu : B(G) \to (X_*(T)^+_{\mathbb{Q}})^\Gamma$$

but we don't really need to work with this concretely. Instead let us state a useful property of this map.

Lemma 3.1. *The map*

$$\nu : |\mathrm{Bun}_\mathrm{G}| \cong \mathrm{B}(\mathrm{G}) \to (\mathrm{X}_*(\mathrm{T})^+_{\mathbb{Q}})^\Gamma$$

is upper semicontinuous (for the topology on $|\mathrm{Bun}_\mathrm{G}|$*).*

Now we come to an important definition.

Definition 3.2. An element $b \in B(G)$ is called basic, if the corresponding slope homomorphism

$$\nu_b \in \mathrm{Hom}_{\breve{E}}(\mathbb{D}_{\breve{E}}, G_{\breve{E}})/G(\breve{E})\text{-conjugacy}$$

factors through the center $Z(G)$.

Let us again look at the case of $G = \mathrm{GL}_n$. The center of GL_n are the diagonal matrices $\lambda \cdot \mathrm{Id}$. This means that there is only one slope occurring in the slope decomposition $V = V^\lambda$, which again corresponds to (V, ϕ), or equivalently the vector bundle $\mathcal{E}(V, \phi)$, being semistable! The following definition and lemma extend the observation just made.

Definition 3.3. A G-bundle $\mathcal{E}$ on X_C is called semistable, if the vector bundle

$$\mathcal{E} \times^{\mathrm{Ad}} \mathfrak{g}$$

is semistable, where

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

is the adjoint representation.

Lemma 3.4. *An element* $b \in B(G)$ *is basic if and only if the corresponding* G*-bundle* $\mathcal{E}_b$ *is semistable.*

3.2. **The Kottwitz point.** The second invariant that we want to use, needs the notion of the algebraic fundamental group. There are several, naturally isomorphic, definitions of it.

Definition 3.5. (1) Let H be a reductive group over $\overline{E}$. As it is necessarily split, we can choose a maximal torus $T \subset H$. Define

$$\pi_1(H, T) = X_*(T)/\text{coroots of } H.$$

Another choice of a maximal torus yields a canonical isomorphism between the two fundamental groups.

(2) Now let G be a reductive group over E. Define

$$\pi_1(G) = \pi_1(G_{\overline{E}}, T)$$

for a choice of a maximal torus T. This carries a $\mathrm{Gal}(\overline{\mathrm{E}}/\mathrm{E})$-action.

The first observation we can make is the following

Lemma 3.6. *Let T be a torus over E. There is a natural isomorphism*

$$\kappa_T : B(T) \to \pi_1(T)_\Gamma.$$

Remark 3.7. Note that if T is a torus, the coroots are trivial and thus $\pi_1(T) = X_*(T)$.

Now let us construct the map

$$\kappa_G : B(G) \to \pi_1(G)_\Gamma$$

or rather we may think of this as a natural transformation $\kappa_{(\cdot)} : B(\cdot) \to \pi_1(\cdot)_\Gamma$. This will be done in several steps. While it is rather involved, the general strategy of proving things involving κ_G will be running step by step through this construction.

Construction 3.8. The construction is done in three steps

(1) First assume that $G = T$ is a torus. Then we can choose the above isomorphism

$$\kappa_T : B(T) \to \pi_1(T)_\Gamma.$$

(2) Now assume that the derived subgroup of G is simply connected and denote the quotient of G by its derived subgroup as D, this is a torus. Then the following commutative diagram

$$\begin{array}{ccc} B(G) & \xrightarrow{\kappa_G} & \pi_1(G)_\Gamma \\ \downarrow & & \| \\ B(D) & \xrightarrow{\kappa_D} & \pi_1(D)_\Gamma \end{array}$$

forces us to define κ_G as the arrow, which makes the diagram commute.

(3) The general case uses the existence of a z-extension, that is a short exact sequence over E

$$1 \to Z \to G' \to G \to 1$$

such that
- Z is a central torus in G',
- the derived subgroup of G' is simply connected.

One precedes to show that the maps

$$B(G') \to B(G)\ ,\ \pi_1(G')_\Gamma \to \pi_1(G)_\Gamma$$

are quotient maps by the actions of $B(Z)$ and $\pi_1(Z)_\Gamma$ and that $\kappa_{G'}$ is equivariant relative to these actions. Thus we get a commutative diagram

$$\begin{array}{ccc} B(G') & \xrightarrow{\kappa_{G'}} & \pi_1(G')_\Gamma \\ \downarrow p & & \downarrow p \\ B(G) & \xrightarrow{\kappa_G} & \pi_1(G)_\Gamma \end{array}$$

and by equivariance there is a unique choice for κ_G.

Furthermore one can show

Lemma 3.9. *The map $\kappa_G : B(G) \to \pi_1(G)_\Gamma$ restricts to an isomorphism*

$$B(G)_{bsc} \to \pi_1(G)_\Gamma$$

where $B(G)_{bsc}$ is the subset of $B(G)$ of basic elements.

Example 3.10. (1) If $G = \mathrm{GL}_n$, then we can compute κ_G by taking the quotient of its derived subgroup $\mathrm{SL_n}$, this can be given as the map

$$\mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m.$$

Thus $\kappa_G : B(\mathrm{GL}_n) \to \pi_1(G)_\Gamma \cong \mathbb{Z}$ sends an isocrystal, or equivalently a vector bundle $\mathcal{E}$, to $\deg(\det(\mathcal{E}))$.

(2) Let us indicate what $B(\mathrm{SL_n})$ looks like. In general for a short exact sequence

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

one has a long exact sequence

$$1 \to G_1(E) \to G_2(E) \to G_3(E) \to B(G_1) \to B(G_2) \to B(G_3) \to 1.$$

We apply this to

$$1 \to \mathrm{SL_n} \to \mathrm{GL_n} \xrightarrow{\det} \mathbb{G}_\mathrm{m} \to 1.$$

Since det is surjective (it actually admits a section), we get

$$B(\mathrm{SL_n}) = \{\mathrm{b} \in \mathrm{B(GL_n)} | \det(\mathrm{b}) = 1\}.$$

So these are just vector bundles $\mathcal{E}$ with $\det(\mathcal{E}) = \wedge^{rk(\mathcal{E})}\mathcal{E} \cong \mathcal{O}$. Note that there is only one vector bundle, which is both semistable and has trivial determinant (namely $\mathcal{O}^{\oplus n}$). This should not come off as a surprise, since this is saying $B(\mathrm{SL_n})_{\mathrm{bsc}} = \{*\}$ and this is a consequence of $\mathrm{SL_n}$ having only a finite center.

Finally, let us relate the Kottwitz and Newton point.

Theorem 3.11. (1) *The map*

$$(\nu, \kappa) : B(G) \to (X_*(T)^+_\mathbb{Q})^\Gamma \times \pi_1(G)_\Gamma$$

is injective.

(2) *There are maps*

$$(X_*(T)^+_\mathbb{Q})^\Gamma \to \pi_1(G)^\Gamma_\mathbb{Q}\ ,\ \pi_1(G)_\Gamma \to \pi_1(G)^\Gamma_\mathbb{Q}$$

and the images of $\kappa(b)$ and $\nu(b)$ agree.

Remark 3.12. In general $\pi_1(G)_\Gamma$ may have torsion. But if it is torsion free, then $\kappa(b)$ is determined by $\nu(b)$.

4. Families of G-bundles

As a first application of the prior section, let us prove the following theorem

Theorem 4.1. *The map*

$$\kappa_G : |\mathrm{Bun_G}| \cong \mathrm{B(G)} \to \pi_1(\mathrm{G})_\Gamma$$

is locally constant.

We will need the following result, which can be proved using affine Grassmanians.

Lemma 4.2. *Let $\widetilde{G} \to G$ be a central extension with kernel a torus. Then*

$$\mathrm{Bun}_{\widetilde{\mathrm{G}}} \to \mathrm{Bun_G}$$

is a surjective map of v-stacks.

In particular this implies that $|\mathrm{Bun}_{\widetilde{\mathrm{G}}}| \to |\mathrm{Bun_G}|$ is a quotient map.

Proof. We will tackle this by going backwards through Construction 3.8.

(1) Let G be arbitrary. Then picking a z-extension we get a commutative diagram

$$\begin{array}{ccc} |\mathrm{Bun}_{\widetilde{G}}| & \twoheadrightarrow & |\mathrm{Bun_G}| \\ \downarrow & & \downarrow \\ \pi_1(\widetilde{G})_\Gamma & \twoheadrightarrow & \pi_1(G)_\Gamma. \end{array}$$

Since the upper horizontal map is a quotient map, local constancy of $\kappa_{\widetilde{G}}$ implies local constancy of κ_G. Thus we have reduced the statement to the case where the derived subgroup is simply connected.

(2) Let G be such that its derived subgroup is simply connected. The quotient $T = G/G_{der}$ is thus a torus. Then we have a commutative diagram

$$\begin{array}{ccc} |\mathrm{Bun_G}| & \longrightarrow & |\mathrm{Bun_T}| \\ \downarrow & & \downarrow \\ \pi_1(G)_\Gamma & = & \pi_1(T)_\Gamma \end{array}$$

Since the above horizontal map is continuous, local constancy of κ_T implies local constancy of κ_G. Thus we have reduced to the case of a torus.

(3) Let T be a torus. Then we can find a surjection $U \to T$ (see [CT87], where U is an induced torus, i.e. a product of the form $\prod_{i=1}^r \mathrm{Res}_{F_i/E}(\mathbb{G}_{m,F_i})$ for F_i/E Galois, and the kernel is again a torus. Thus this is a z-extension and we are reduced to the case, where T is an induced torus.

(4) Let T be an induced torus. By Shapiro's Lemma $\pi_1(T)_\Gamma$ is torsionfree. But by Remark 3.12. this means that $\kappa(b)$ is determined by $\nu(b)$. By 3.1 ν is semicontinuous. But if T is a torus this is equivalent to local constancy (the ordering on $(X_*(T)^+_{\mathbb{Q}})^\Gamma$ becomes trivial). Combining these two prior observations, we get that κ is locally constant.

□

4.1. **Geometrically fibrewise trivial G-bundles.** For general $S \in \mathrm{Perf_k}$ we think of X_S as a family of curves over $s \in S$. Let us define

$$\mathrm{Bun_G^1} \subset \mathrm{Bun_G}$$

to be the substack of geometrically fiberwise trivial G-bundles. So if we consider a G-bundle ω as $\otimes$-exact functor, then this means that the composition

$$\mathrm{Rep_E(G)} \xrightarrow{\omega} \mathrm{Bun(X_S)} \xrightarrow{i^*} \mathrm{Bun(X_{\bar{s}})}$$

is isomorphic to the trivial fiber functor. Here i^* is the pullback via $i : X_{\bar{s}} \to X_S$. An example of such a G-bundle is given by the constant torsor $\mathcal{E}_1$, this can be also seen as the morphism of stacks

$$x_1 : * \to \mathrm{Bun_G^1} \subset \mathrm{Bun_G}.$$

Concretely, for $S \in \mathrm{Perf_k}$ this is the fiber functor

$$\mathrm{Rep_E(G)} \xrightarrow{\mathrm{N_1}} \mathrm{Isoc} \to \mathrm{Bun(X_S)} \tag{4.1}$$

where the first functor comes from 2.4 and the latter from Equation (1.1).
Now let us compute

$$* \times_{\mathrm{Bun_G^1}} *.$$

This takes $S \in \mathrm{Perf_k}$ to the automorphisms of the trivial G-bundle Equation (4.1). But the automorphisms of the trivial bundle are just

$$G(H^0(X_S, \mathcal{O}_{X_S})) = G(\underline{E}(S)) = \underline{G(E)}(S).$$

So we can descend x_1 to the classifying stack of pro-étale $\underline{G(E)}$-torsors

$$[x_1] : [*/\underline{G(E)}] \to \mathrm{Bun_G^1}$$

which brings us to the main theorem of this section:

Theorem 4.3. (1) *The substack* $\mathrm{Bun_G^1} \subset \mathrm{Bun_G}$ *is open.*
(2) *The above map* $[*/\underline{G(E)}] \to \mathrm{Bun_G^1}$ *is an isomorphism.*

To prove this theorem, we need to settle the GL_n case:

Theorem 4.4. *Let S be as above. Then the category of pro-étale local systems $\mathbb{L}$ on S is equivalent to the category of vector bundles on X_S whose Harder-Narasimhan filtration is constant* 0 *(i.e. geometrically fiberwise trivial), via $\mathbb{L} \mapsto \mathbb{L} \otimes_{\underline{E}} \mathcal{O}_{X_S}$.*

Remark 4.5. This condition on the Harder-Narasimhan filtration being 0 is also equivalent to the vanishing of $\nu : S \to (X_*(T)^+_{\mathbb{Q}})^{\Gamma}$.

Proof of Theorem 4.3.

(1) To show that $\mathrm{Bun}^1_{\mathrm{G}} \subset \mathrm{Bun}_{\mathrm{G}}$ is open, we need to show that for a G-bundle $\mathcal{E} \in \mathrm{Bun}_{\mathrm{G}}(\mathrm{S})$ the subset of $|S|$ where $\mathcal{E}_{\bar{x}}$ is trivial is open.
Since $S \in \mathrm{Perf_k}$, it admits a cover by qcqs opens and thus we can assume S is qcqs. Furthermore, if $T \to S$ is surjective and T qcqs, then $|T| \to |S|$ is a quotient map. Thus we may assume that S is strictly totally disconnected. By Lemma 3.1 the locus where the Newton point of $\mathcal{E}$ vanishes is open (since 0 is a minimal element). Since this contains the locus where the geometric fibres are trivial, we can assume that the Newton point is 0 on S. Thus for any representation $r : G \to \mathrm{GL}(V)$ the vector bundle $\mathcal{E} \times^r (V \otimes X_S)$ on X_S has trivial geometric fibers. Applying 4.4 this is equivalent to an $\underline{E}$-local system over S. But S is strictly totally disconnected and thus any such local system is trivial (see the lemma below) and thus local systems are equivalent to free modules over $C^0(|S|, E)$, i.e. we get a G-torsor over $\mathrm{Spec}(C^0(|S|, E)$
$$\omega : \mathrm{Rep_E(G)} \to \mathrm{Bun^1(X_S)} \cong \mathrm{Bun(Spec(C^0(|S|, E))}.$$
Now given $s \in \pi_0(S)$ the local ring $\varinjlim_{s \in U} C^0(|U|, E)$ is henselian, since it is the local ring of the analytic adic space $\underline{\pi_0(S)} \times \mathrm{Spa}(E)$. Now a G-torsor over a Henselian ring that is trivial at its closed point is actually trivial (since G is smooth), i.e. being trivial is a local property.

(2) Let's begin by giving an explicit description of
$$[x_1](S) : [*/\underline{G(E)}](S) \to \mathrm{Bun^1_G(S)}.$$
Let $(V, r) \in \mathrm{Rep_E(G)}$ and $\mathcal{P}$ a pro-étale $\underline{G(E)}$-torsor. Then one can form the pro-étale local system $\mathcal{P} \times^r \underline{V}$. By the above theorem this is equivalent to a vector bundle $\mathcal{E}_r$ on X_S which is geometrically fiberwise trivial. Thus we get a G-torsor on X_S
$$\omega_{\mathcal{P}} : \mathrm{Rep_E(G)} \to \mathrm{Bun(X_S)}$$
$$(V, r) \mapsto \mathcal{E}_r.$$
Now to get the converse construction, start with a geometrically fiberwise trivial G-torsor $\mathcal{E}$. Define the sheaf on $\mathrm{Perf_S}$
$$\mathrm{Isom}(\mathcal{E}, \mathcal{E}_1).$$
This carries an action of $\mathrm{Aut}(\mathcal{E}_1) = \underline{\mathrm{G(E)}}$. Once we know that $\mathcal{E}$ is pro-étale locally trivial on S, then this shows that we get a torsor. But the argument in (1) shows that being pointwise trivial implies being trivial in an neighbourhood (after passing to a cover by a strictly totally disconnected space).

□

Let us finish with the result mentioned in the above proof:

Lemma 4.6. *Let S be a strictly totally disconnected perfectoid space. Then any pro-étale $\underline{G(E)}$-torsor on S is trivial.*

One applies this result to the $G = \mathrm{GL}_n$ case to get the desired conclusion (namely that the local systems are trivial).

5. The semistable case via G_b

In the next step we want to go from the trivial case $b = 1$ to the basic case $b \in B(G)_{bsc}$. The heuristics that we want to make precise is that the basic case is equivalent to the trivial case "up to a twist".

Definition 5.1. Let $b \in B(G)$. Choosing a representative in $G(\breve{E})$, define for an E-algebra R

$$G_b(R) := \{g \in G(\breve{E} \otimes_E R) | g = b\sigma(g)b^{-1}\}.$$

This is representable by a reductive group over E. A different choice of a representative yields a group which is isomorphic via an inner automorphism.

This group is also sometimes denoted as J_b. Let us collect a few properties of this group.

Lemma 5.2. *(1) $G_b \times_E \breve{E}$ is isomorphic to the centralizer of the slope homomorphism $\nu_b : \mathbb{D} \to G \times_E \breve{E}$,*
(2) If G is quasi-split, it is an inner form of a Levi subgroup of G,
(3) G_b is an inner form of G if and only if b is basic.

For this section we will make use of the third part of this lemma. The other two will allow us to generalize to the non-basic case. Now let us look at the geometric side of twisting. For this, we will apply the following Proposition

Proposition 5.3. *Let X be a topos, H a group in X and T an H-torsor. Write $H^T := \underline{\mathrm{Aut}}(T)$ as a group in X. This is also called the "pure inner twisting" of H by T. The morphism of stacks of H (resp. H^T) torsors*

$$[*/H] \to [*/H^T]$$
$$S \mapsto \underline{\mathrm{Isom}}(S, T)$$

is an equivalence.

For us the topos is given by the sheaves on the étale site of X_S, H is the group $G \times_{\mathrm{Spa}(E)} X_S$ and the torsor will be $\mathcal{E}_b \to X_S$. Then the "geometric" twist of $G \times_{\mathrm{Spa}(E)} X_S = \mathrm{Aut}(\mathcal{E}_1)$ is $\mathrm{Aut}(\mathcal{E}_\mathrm{b})$. Note that $\underline{\mathrm{Aut}}(\mathcal{E}_b)$ is a slightly ambiguous notion. Usually we mean by this the sheaf on $\mathrm{Perf_k}$, but now we consider it as a sheaf on the étale site of X_S (i.e. as done classically). So to recap, given $b \in B(G)$ we have the twist G_b and the twist $\underline{\mathrm{Aut}}(\mathcal{E}_b)$. The next proposition tells us that these are compatible with each other. For this we will use the description of torsors as sheaves.

Proposition 5.4. *Let $S \in \mathrm{Perf_k}$, $b \in B(G)$ basic and $\mathcal{E}_b \to X_S$ the associated étale G-torsor (see also Equation (4.1)). Then the sheaf of groups $G_b \times_{Spa(E)} X_S$ is the pure inner twisting of $G \times_{Spa(E)} X_S$ by $\mathcal{E}_b$, i.e.*

$$\underline{\mathrm{Aut}}(\mathcal{E}_b) \cong G_b \times_{Spa(E)} X_S.$$

Proof. We start by describing $\mathcal{E}_b$ as

$$\mathcal{E}_b = (G_{\breve{E}} \times_{\mathrm{Spa}(\breve{E})} Y_S)/((b\sigma) \otimes \varphi)^{\mathbb{Z}}.$$

This admits a right action by the group $G \times X_S = (G_{\breve{E}} \times_{\mathrm{Spa}(\breve{E})} Y_S)/(\mathrm{Id} \times \varphi)^{\mathbb{Z}}$ making it a torsor for this group. Now the group $G_b \times X_S = (G_b \times Y_S)/(\mathrm{Id} \times \varphi)^{\mathbb{Z}}$ acts on this torsor on the left and the action is compatible with the $G \times X_S$-torsor structure (since one acts on the left and the other on the right). Thus we get a morphism

$$G_b \times X_S \to \underline{\mathrm{Aut}}(\mathcal{E}_b).$$

It suffices to check that this is an isomorphism after pulling back by the étale cover $Y_S \to X_S$. Then evaluating at $T = \mathrm{Spa}(R, R^+) \to Y_S$ affinoid sous-perfectoid, this becomes the map

$$G_b(R) = \{g \in G(\breve{E} \otimes_E R) | g = b\sigma(g)b^{-1}\} \to \underline{\mathrm{Aut}}(\mathcal{E}_b)(R) \cong G(R)$$

where we have used the $\breve{E}$ algebra structure of R to trivialize $\underline{\mathrm{Aut}}(\mathcal{E}_b)$ (since over Y_S it is the trivial torsor). Since b is basic, we have by 5.2 that

$$G_b \times_E \breve{E} \to G \times_E \breve{E}$$

is an isomorphism and we can naturally identify the above map with this. □

So now we have verified that both our notions of twisting agree. Applying 5.3 we get

Corollary 5.5. *For b basic there is an isomorphism of v-stacks*

$$\mathrm{Bun_G} \cong \mathrm{Bun_{G_b}}$$

that induces an isomorphism $\mathrm{Bun_G^b} \cong \mathrm{Bun_{G_b}^1}$.

Let us finish the semistable case by the following description of the semistable locus.

Theorem 5.6. (1) *The semistable locus*

$$\mathrm{Bun}_G^{ss} \subset \mathrm{Bun}_G,$$

is open

(2) *There is a canonical decomposition as open/closed substacks*

$$\mathrm{Bun}_G^{ss} = \coprod_{b \in B(G)_{bsc}} \mathrm{Bun}_G^b,$$

(3) *For b basic there is an isomorphism*

$$[*/\underline{G_b(E)}] \cong \mathrm{Bun}_G^b.$$

Proof. (1) For the first statement we want to apply 3.1. For this notice that Bun_G^{ss} is the preimage of $\nu(b)$ under ν for b basic. Thus the result from the fact that $\nu(b)$ for basic b are minimal with respect to the order.

(2) By 3.9 we know that $\kappa_G : |\mathrm{Bun}_G^{ss}| \cong B(G)_{bsc} \to \pi_1(G)_\Gamma$ is an isomorphism and by 3.1 it is continuous. Thus each component in the decomposition

$$\mathrm{Bun}_G^{ss} = \coprod_{b \in B(G)_{bsc}} \mathrm{Bun}_G^b$$

is open.

(3) Now applying 5.5 we know that for b basic $\mathrm{Bun}_G^b \cong \mathrm{Bun}_{G_b}^1$ and applying 4.3 to G_b, we get

$$\mathrm{Bun}_G^b \cong \mathrm{Bun}_{G_b}^1 \cong [*/\underline{G_b(E)}].$$

□

6. Between semistable and non-semistable

Now that we covered the semistable case, the question is how to generalize this result. Let us start by giving the following example.

Example 6.1. Let's take $C \in \mathrm{Perf_k}$ complete and algebraically closed. Then on X_C we have the vector bundle $\mathcal{E}_b = \mathcal{O} \oplus \mathcal{O}(1)$. This corresponds to $b = \begin{bmatrix} 1 & 0 \\ 0 & \pi^{-1}. \end{bmatrix} \in B(\mathrm{GL}_2)$. What are the automorphisms of this? Of course $\mathrm{Aut}(\mathcal{O}) \times \mathrm{Aut}(\mathcal{O}(1)) \cong \underline{E}^\times \times \underline{E}^\times$ is a subgroup of the automorphisms of $\mathcal{E}_b$. But we have additional automorphisms. Namely, homomorphism $f : \mathcal{O} \to \mathcal{O}(1)$ yields also an automorphism by sending $(a, b) \in \mathcal{O} \oplus \mathcal{O}(1)$ to $(a, b + f(a))$. Thus the Banach-Colmez space $\mathcal{BC}(\mathcal{O}(1))$ also yields automorphisms and combining everything together we see

$$\mathrm{Aut}(\mathcal{E}_b) \cong \begin{bmatrix} \underline{E}^\times & \mathcal{BC}(\mathcal{O}(1)) \\ 0 & \underline{E}^\times. \end{bmatrix}$$

Thus this example shows that the general case requires more work. To give a description of the non-semistable case, we first look at automorphisms which preserve the grading. This will be similar to the basic case. In the prior example this will "extract" $\underline{E}^\times \times \underline{E}^\times$. Then we can generalize to the non semistable case with the help of Banach-Colmez spaces.

Definition 6.2. *Let*$S \in \mathrm{Perf_k}$.

(1) The category $\mathrm{Gr}^{\mathbb{Q}}\mathrm{Bun}(X_S)$ has as objects $\mathbb{Q}$-graded vector bundles on X_S and morphisms preserving the grading.

(2) The functor $\mathrm{Bun}_G^{HN-split}$ takes S to the groupoid of $\otimes$-exact functors $\mathrm{Rep_E(G)} \to \mathrm{Gr}^{\mathbb{Q}}\mathrm{Bun}(X_S)$ such that for $\mathcal{E} = \bigoplus \mathcal{E}^\lambda$, the subbundle $\mathcal{E}^\lambda$ is semistable of slope λ.

Example 6.3. Take $b \in B(G)$. Then the torsor $\mathcal{E}_b \in \mathrm{Bun_G(S)}$

$$\mathrm{Rep_E(G)} \xrightarrow{N_b} \mathrm{Isoc} \to \mathrm{Bun(X_S)}$$

can be refined to a torsor $\mathcal{E}_b^{gr} \in \mathrm{Gr}^{\mathbb{Q}}\mathrm{Bun_G(S)}$ by remembering the $\mathbb{Q}$-grading coming from Isoc.

We come to the main theorem of this section

Theorem 6.4. *Let $b \in B(G)$ and $S \in \mathrm{Perf_k}$ affinoid.*

(1) *There is a natural map*

$$G_b \times_E X_S^{alg} \to \underline{\mathrm{Aut}}(\mathcal{E}_b^{gr})$$

considered as group schemes over X_S^{alg}*, which is an isomorphism.*

(2) *The natural map*

$$\coprod_{b \in B(G)} [*/\underline{G_b(E)}] \to \mathrm{Bun_G^{HN-split}}$$

is an isomorphism.

Proof. (1) The proof of this part is similar to 5.4. In fact the first part of that proof did not use that b is basic and thus we can apply the same argument to get a map

$$G_b \times_E X_S \to \underline{\mathrm{Aut}}(\mathcal{E}_b). \tag{6.1}$$

Using the Tannakian description one can see that it factors through $\mathrm{Aut}(\mathcal{E}_\mathrm{b}^\mathrm{gr})$ as it already acts on the isocrystals. Now we can again pullback by an étale cover to check that this is an isomorphism. If R is endowed with an $\breve{E}$-algebra structure the above map gets identified with

$$G_b(R) \to \underline{\mathrm{Aut}}(\mathcal{E}_b^{gr})(R) \cong \mathrm{Cent}(\nu_b)(R)$$

where $\nu_b : \mathbb{D} \to G \times_E \breve{E}$ is the Newton point of b. To see the above isomorphism, one reduces via Tannakian formalism to the GL_n-case, which we will sketch now:
Let $V = \bigoplus_{i=1}^n V^{\lambda_i}$ be an isocrystal, where the grading corresponds to a homomorphism $\nu_b : \mathbb{G}_m \to \mathrm{GL}_n \times_E \breve{E}$ (up to scaling since the slopes could be any rational). Given $g \in \mathrm{GL}_n(R)$, it is in $\underline{\mathrm{Aut}}(\mathcal{E}_b^{gr})(R)$ if and only if $g \cdot V^{\lambda_i} \subset V^{\lambda_i}$ for all $1 \leq i \leq n$. But the latter condition means exactly

$$\nu_b(z)(gv_i) = z^i \cdot gv_i = g \cdot (z^i v_i) = g(\nu_b(z)v_i).$$

Doing this for all $1 \leq i \leq n$, we see that indeed g commutes with $\nu_b(z)$, i.e. it lies in the centralizer.
But by 5.2 $G_b \times_E \breve{E}$ is isomorphic $\mathrm{Cent}(\nu_b)$ and thus we get the desired isomorphism.

(2) Let us prove surjectivity. We may assume S is strictly totally disconnected and $\mathcal{E}^{gr} \in \mathrm{Bun_G^{HN-split}}(\mathrm{S})$. For any (geometric) point $s \in S$, there is some $b \in B(G)$ such that $(\mathcal{E}^{gr})_s \cong (\mathcal{E}_b^{gr})_s$. Now the $\mathbb{Q}$-filtration is locally constant, so we can find an open neighbourhood such that

$$\underline{\mathrm{Isom}}(\mathcal{E}_b^{gr}, \mathcal{E}^{gr})$$

defines an $\underline{\mathrm{Aut}}(\mathcal{E}_b^{gr})$-torsor over X_S^{alg}. (To find that neighbourhood one can use Tannakian formalism and that a faithful representation $\rho : G \to \mathrm{GL}_n$ induces a map $B(G) \to B(\mathrm{GL}_n)$ with finite fibers, see [RR96, Proposition 2.4].) Thus we get a map

$$S \to \mathrm{Bun_{G_b}}$$

and s lands in $\mathrm{Bun_{G_b}^1}$. By 4.3 we can replace S by an open neighbourhood and then 4.6 allows us to conclude that the above torsor is trivial.

□

7. Non-semistable points

Our aim now is to look at the v-sheaf

$$\widetilde{G}_b := \underline{\mathrm{Aut}}(\mathcal{E}_b).$$

For any representation $(V, r) \in \mathrm{Rep_E(G)}$ write $(r_*\mathcal{E}_b)^{\geq \lambda}$ for the Harder-Narasimhan filtration. Now for any $\lambda > 0$ define the group

$$\widetilde{G}_b^{\geq \lambda} \subset \widetilde{G}_b$$

as the subgroup of automorphisms $\gamma : \mathcal{E}_b \to \mathcal{E}_b$ such that

$$(\gamma - 1)(r_*\mathcal{E}_b)^{\geq \lambda'} \subset (r_*\mathcal{E}_b)^{\geq \lambda' + \lambda}$$

for all λ' and $(V, r) \in \mathrm{Rep_E(G)}$. Also we can define

$$\widetilde{G}_b^{> \lambda} = \bigcup_{\lambda' > \lambda} \widetilde{G}_b^{\geq \lambda'}$$

which eventually becomes constant. The automorphism group of $N_b : \mathrm{Rep_E(G)} \to \mathrm{Isoc}$ is $G_b(E)$ and thus we get an embedding

$$\underline{G_b(E)} \hookrightarrow \widetilde{G}_b.$$

On the the other hand assume we have any $\gamma \in \underline{\mathrm{Aut}}(\mathcal{E}_b)$ and any representation $(V, r) \in \mathrm{Rep_E(G)}$ then γ induces an isomorphism $[\gamma]$ of the $\mathbb{Q}$-graded bundle

$$\bigoplus_{\lambda \in \mathbb{Q}} Gr^\lambda(r_*\mathcal{E}_b) = \bigoplus_{\lambda \in \mathbb{Q}} (r_*\mathcal{E}_b)^{\geq \lambda}/(r_*\mathcal{E}_b)^{>\lambda} \tag{7.1}$$

and by 6.4 we can regard $[\gamma]$ as an element in $\underline{G_b(E)}$ and thus we get a section of the above embedding. To summarize, we have an isomorphism

$$\widetilde{G}_b \cong \widetilde{G}_b^{>0} \rtimes \underline{G_b(E)}.$$

The following result makes use of the formalism of filtered and graded fiber functors, as an extension of the usual Tannakian formalism.

Proposition 7.1. *With the above notation for any $\lambda > 0$ we have an isomorphism*

$$\widetilde{G}_b^{\geq \lambda}/\widetilde{G}_b^{>\lambda} \xrightarrow{\sim} \mathcal{BC}((Ad\mathcal{E}_b)^{\geq \lambda}/(Ad\mathcal{E}_b)^{>\lambda})$$

the Banach-Colmez space associated to the slope $-\lambda$ part of the isocrystal $(Lie(G) \otimes_E \breve{E}, Ad(b)\sigma)$.

Thus $\widetilde{G}_b$ is an extension of $\underline{G_b(E)}$ by a successive extension of positive Banach-Colmez spaces and as a consequence $\widetilde{G}_b \to *$ is representable in locally spatial diamonds of dimension $\langle 2\rho, \nu_b \rangle$ (where ρ is the half-sum of the positive roots).

If G is quasi-split, it comes equipped with $A \subset T \subset B$, a maximal split torus A, a maximal torus T and a Borel B. Then for any b, we can find a σ-conjugacy class, such that $\nu_b : \mathbb{D} \to G$ factors through A. Then we get

- M_b the centralizer of ν_b
- $B \subset P_b^+$ the associated parabolic and P_b^- its opposite
- b_M which is the equivalence class of b, but now regarded as element in $M_b(\breve{E})$.

Then we can define the X_R^{alg} group schemes

$$Q := \mathcal{E}_{b_M} \times^{M_b} P_b^-,$$
$$R_uQ := \mathcal{E}_{b_M} \times^{M_b} R_uP_b^-$$

and one can give a concrete description

$$\widetilde{G}_b(R, R^+) = Q(X_R^{alg}),$$
$$\widetilde{G}_b^{>0}(R, R^+) = R_uQ(X_R^{alg}).$$

We can finish with the following Proposition

Proposition 7.2. *Let $b \in B(G)$. The map*

$$x_b : * \to \mathrm{Bun_G^b}$$

is a surjective map of v-stacks and induces an isomorphism

$$[x_b] : [*/\widetilde{G}_b] \xrightarrow{\sim} \mathrm{Bun_G^b}.$$

As a consequence we get a map with a splitting

$$\mathrm{Bun_G^b} \cong [*/\widetilde{\mathrm{G}}_\mathrm{b}] \to [*/\underline{\mathrm{G_b(E)}}].$$

Sketch of proof. Let $\mathcal{E} \in \mathrm{Bun_G^b(S)}$ and we can assume that S is strictly totally disconnected.

- Using the Harder-Narasimhan filtration one can define an $H^{\geq 0} = \underline{\mathrm{Aut}}_{\mathrm{filtered}}(\mathcal{E}_b)$ torsor
$$T = \underline{\mathrm{Isom}}_{\mathrm{filtered}}(\mathcal{E}_b, \mathcal{E})$$
which is a reduction of the $H = \underline{\mathrm{Aut}}_{\mathrm{filtered}}(\mathcal{E}_b)$ torsor $\underline{\mathrm{Isom}}(\mathcal{E}_b, \mathcal{E})$. Thus it suffices to trivialize the torsor T.
- The image of the torsor $[T] \in H^1_{\text{ét}}(X_R^{alg}, H^{\geq 0})$ in $H^1_{\text{ét}}(X_R^{alg}, H^{\geq 0}/H^{>0})$ is locally trivial, since it parametrizes isomorphisms of the graded fiber functors, which are the semisimplification of the filtered fiber functors, see also (7.1).

- One concludes by using that $H^1_{\text{ét}}(X_R^{alg}, H^{>0}) = 0$ and the associated long exact sequence induced by $1 \to H^{>0} \to H^{\geq 0} \to H^{\geq 0}/H^{>0} \to 1$.

□

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