NA - Geometry talks

London '21

1. The Tate algebra

Let K be a complete field w.r.t. a non-Archimedian absolute value that is non trivial

Definition

For $n \geq 1$ define the unit ball

$$\boxed{B^n(\overline{K}) := \left\{ (x_1, \dots, x_n) \in \overline{K}^n ;\ |x_i| \leq 1 \right\}}$$

in $\overline{K}^n$

Lemma A formal powerseries $f = \sum_{v \in \mathbb{N}^n} c_v X^v \in K[[X_1, \dots, X_n]]$, $X^v = X_1^{v_1} \cdots X_n^{v_n}$ converges on $B^n(\overline{K})$

$$\iff \lim_{|v| \to \infty} |c_v| = 0$$

▷ Proof. if f converges at $(1, \dots, 1)$ then $\sum_v c_v$ converges $\Rightarrow \lim_{v \to \infty} |c_v| = 0$

conversely, if $x \in B^n(\overline{K})$ then there ex. a finite ext. $L|K$ st x_i belong to L (complete)

if $|c_v| \to 0$ $(|v| \to \infty)$ then $\lim_{|v| \to \infty} |c_v| |x^v| = 0$, thus $f(x)$ defines a Cauchy sequence in L (complete)

this enlightens the following definition:

Definition

we define the Tate algebra of restricted power series

$$\boxed{\begin{array}{l} \text{The } K \text{ algebra } T_n := K\langle X_1, \dots, X_n \rangle \text{ consisting of elements} \\ \qquad f = \sum_{v \in \mathbb{N}^n} c_v X^v \quad \text{st. } |c_v| \xrightarrow{|v| \to \infty} 0 \ , \ f \in K[[X_1, \dots, X_n]] \ , \ c_v \in K \end{array}}$$

Gauß norm: $T_n \ni f : \ |f| = \max_v |c_v| \ , \ f = \sum c_v X^v$

Proposition

T_n is complete with respect to the Gauß norm $|\cdot|$, i.e. $\sum_{i=0}^{\infty} f_i$ converges w.r.t. restricted p.s. $f_i = \sum_v c_{iv} X^v \in T_n$ if $\lim_{i \to \infty} f_i = 0$

▷

Proposition (Maximum principle) Let $f \in T_n$. Then $|f(x)| \leq |f|$ for all points $x \in B^n(\bar{K})$ and $\exists\, x \in B^n(\bar{K})$ st $|f(x)| = |f|$

▷ Proof. wlog $|f| = 1$ and consider the can. epi $\pi : R\langle X_1, \dots, X_n\rangle \twoheadrightarrow k[X_1, \dots, X_n]$
$f \longmapsto \bar{f} \neq 0$

$$\begin{array}{ccccc} k & \twoheadleftarrow & \overset{R}{\overset{\shortparallel}{\mathcal{O}_K}} & \subseteq & K \\ & & | & & | \\ \bar{k} & \twoheadleftarrow & \bar{R} & \subseteq & \bar{K} \end{array}$$

$\Rightarrow \exists\, \bar{x} \in \bar{k}^n : \bar{f}(\bar{x}) \neq 0$.

Have comm. diagram

$$\begin{tikzcd}
f \arrow[rrr, bend left] \arrow[d] & R\langle X_1, \dots, X_n\rangle \arrow[r, "\pi"] \arrow[d, "ev_x"] & k[X_1, \dots, X_n] \arrow[d, "ev_{\bar{x}}"] & \bar{f} \arrow[d] \\
f(x) \arrow[rrr] & \bar{R} \arrow[r] & \bar{k} & \bar{f}(\bar{x}) \neq 0
\end{tikzcd}$$

$\Rightarrow |f(x)| = 1$

Theorem (Weierstraß division) see [2] 2.2 15 ff.

A restricted power series $g = \sum g_\nu X_n^\nu \in T_n$ w/ coefficients $g_\nu \in T_{n-1}$ is called X_n distinguished of some order $n \in \mathbb{N}$ if
- g_s is a unit in T_{n-1}
- $|g_s| = |g|$ and $|g_s| > |g_\nu|$ for $\nu > s$.

Let $g \in T_n$ be X_n distinguished of some order s. Then for any $f \in T_n$ there ex a unique series $q \in T_n$ and a polyn. $r \in T_{n-1}[X_n]$ of $\deg r < s$ such that

$$f = qg + r$$

furthermore, $|f| = \max\{|q|, |g|, |r|\}$

Theorem (Noether normalization)

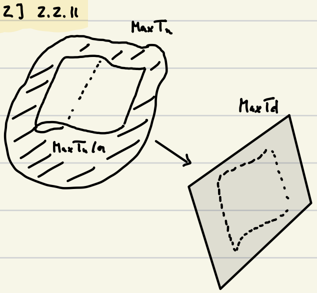
For any proper ideal $\mathfrak{a} \subsetneq T_n$ there ex. a K algebra monomorphism $T_d \longrightarrow T_n$, some $d \geq 0$, such that the composition

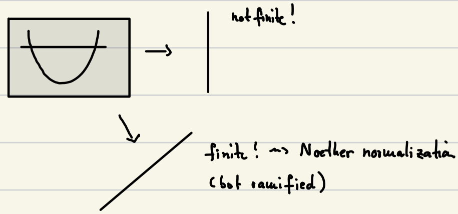
$$\begin{tikzcd} T_d \arrow[r] \arrow[rr, bend right, dashed] & T_n \arrow[r] & T_n/\mathfrak{a} \end{tikzcd}$$

is a finite monomorphism

Proof. [2] 2.2.11

picture:





$$\begin{tikzcd} \underset{\text{field}}{\underbrace{T_d}} \arrow[r] \arrow[rr, bend left] & T_n \arrow[r] & \underset{\text{field}}{\underbrace{T_n/\mathfrak{m}}} \end{tikzcd}$$

Consequences of N.n.:

$B^n(\overline{K}) \twoheadrightarrow \operatorname{Max} T_n \qquad x \longmapsto \mathfrak{m}_x = \{h \in T_n \mid h(x) = 0\}$

▷ $f \in T_n$: $\quad ev_x : T_n \twoheadrightarrow K(x)$ cont. epi kernel $\mathfrak{m}_x$ maximal

if $\mathfrak{m}_x$ is maximal Then $\underbrace{T_n \longrightarrow T_n/\mathfrak{m}_x \hookrightarrow \overline{K}}_{\varphi_x}$: This map is contractive (note that $T_n/\mathfrak{m}_x$ is finite)

$|\varphi(a)| \leq |a|$ $a \in T_n$: assume $\exists\, a \in T_n$ st $|\varphi(a)| > |a|$. Then $a \neq 0$ and w.l.g. $|a| = 1$. Define $\alpha := \varphi(a)$ write $p(t) = t^r + c_1 t^{r-1} + \dots + c_r \in K[t]$ as as minimal polynomial of α over K. If $\alpha_1, .., \alpha_r$ are the conjugate elements of this α in $\overline{K}$, $p(t) = \prod_{j=1}^{r}(t - \alpha_j)$ and $K(\alpha) \cong K(\alpha_j)$, thus $|\alpha_j| = |\alpha|$ (all j)

now $|p(0)| = |c_r| = \prod_{j=1}^{r} |\alpha_j| = |\alpha|^r$, also $|c_j| \leq |\alpha|^j \underset{|\alpha| \geq 1}{\leq} |\alpha|^r = c_r$

$\Rightarrow \quad p(a) = \sum_{i=0}^{r} c_i a^{r-i}$ is a unit in T_n

characterization of units in T_n: $f \in T_n$ is a unit $\Leftrightarrow |f - f(0)| < |f(0)|$ [2] 3.1.4.

$\Rightarrow$ must be mapped to a unit in K' under φ, but on the other hand, $\varphi(p(a)) = p(\varphi(a)) = p(\alpha) = 0$ ↯ $\Rightarrow |\varphi(a)| \leq |a|$ $\forall\, a \in T_n \Rightarrow \varphi : T_n \to \overline{K}$ continuous

now set $y_i = \varphi(X_i)$ $i = 1, .., n$ Then $y = (y_1, .., y_n)$ belongs to $B^n(\overline{K})$ and φ coincides with φ_x thus $\mathfrak{m} = \mathfrak{m}_{(y_1, .., y_n)} = \{h \in T_n \mid h(y) = 0\}$

<u>Proposition</u> T_n is Noetherian, i.e. every ideal $\mathfrak{a} \subseteq T_n$ is finitely generated

▷ by induction: assume T_{n-1} is Noetherian, $\mathfrak{a} \subseteq T_n$ non-trivial ideal.

if $g \in \mathfrak{a}, g \neq 0$, Weierstraß division gives that $T_n/(g)$ is a finite T_{n-1}-module, thus a Noetherian T_{n-1} module because T_{n-1} is Noetherian, thus $\mathfrak{a}/(g)$ is fin. generated over T_{n-1}, so $\mathfrak{a}$ is fin generated over T_n //

<u>Further properties</u>:

T_n is factorial, hence normal

T_n is Jacobson $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{m} \ni \mathfrak{a}} \mathfrak{m}$

every max ideal $\mathfrak{m} \subseteq T_n$ satisfies $ht(\mathfrak{m}) = n \Rightarrow \dim T_n = n$ and can be generated by n elements

<u>Ideals in Tate algebras</u>

$\mathfrak{a} \subseteq T_n$, then

- $\mathfrak{a}$ is complete, hence closed
- $\mathfrak{a}$ is strictly closed: $\forall f \in T_n : \exists a_0 \in \mathfrak{a} : |f - a_0| = \inf_{a \in \mathfrak{a}} |f - a|$

$\exists$ generators $a_1, \dots, a_r$ of $\mathfrak{a}$ st $|a_i| = 1$ $f \in \mathfrak{a} : \exists f_1, \dots, f_r$ st $f = \sum_{i=1}^{r} f_i a_i$
$|f_i| \leq |f|$

[2] 2.3.7

$B^n(\overline{K}) \twoheadrightarrow \mathrm{Max}\, T_n$

Affinoid algebras

viewed $f \in T_n$ as functions $B^n(\overline{K}) \to \overline{K}$. For $\mathfrak{a} \subseteq T_n$ consider the zero set $V(\mathfrak{a}) = \{ x \in B^n(\overline{K}) : f(x) = 0 \ \forall f \in \mathfrak{a} \} \subseteq B^n(\overline{K})$

Restrict fcts from $B^n(\overline{K}) \to V(\mathfrak{a})$

$\rightsquigarrow$ get map $T_n \longrightarrow \{ \text{functions on } V(\mathfrak{a}) \to \overline{K} \}$ vanishing on $\mathfrak{a}$

$\Rightarrow$ $T_n / \mathfrak{a}$ is algebra of functions vanishing on $V(\mathfrak{a})$

Definition a K algebra A is called affinoid if $\exists$ epi of K algebras $\alpha : T_n \twoheadrightarrow A$ for some $n \geq 0$

$\hookrightarrow$ form category with K algebras homom. as morphisms

Properties of affinoid algebras:

immediate consequences of properties of T_n:

- A is Noetherian, Jacobson and satisfies Noether-normalization
  $\rightsquigarrow$ follows from fact that these properties behave well under K-algebra quotients.
- if A is affinoid, $\mathfrak{q} \subseteq A$ st $\mathrm{rad}\,\mathfrak{q}$ is max., then $A/\mathfrak{q}$ is fin dim v.s. over K: choose $d \geq 0$ st $T_d \xhookrightarrow[\varphi]{\text{fin.}} A/\mathfrak{q} \twoheadrightarrow A/\mathfrak{m}$
  $\Rightarrow$ $T_d \to A/\mathfrak{m}$ is finite and $\varphi(T_d)$ contains no nilpotents, thus $T_d \hookrightarrow A/\mathfrak{m}$ = field. $\Rightarrow$ T_d field $\Rightarrow d = 0$ $\Rightarrow$ $A/\mathfrak{q}$ fin dim K vs. [φ was finite morph.]

Topology on affinoid algebras

Definition we endow A w/ a "residue norm" $|\cdot|_\alpha$ given by $\alpha : T_n \twoheadrightarrow A$

$$\boxed{|\alpha(f)|_\alpha = \inf_{a \in \ker f} |f - a|}$$

that satisfies the following properties: $|\cdot|_\alpha$ is K-algebra norm and induces the quotient topology on A, $\alpha : T_n \twoheadrightarrow A$ is continuous and open; Furthermore, A is complete under $|\cdot|_\alpha$ and for $\bar{f} \in A$ $\exists$ lift $f \in A$ st. $|f| = |\bar{f}|_\alpha$ [because $\mathfrak{a} \subseteq T_n$ are strictly closed]. But: topology depends on α!

Viewing elements f of an affinoid K-algebra $T_n/\mathfrak{a}$ as $\overline{K}$-valued functions on $V(\mathfrak{a})$, we can define a "sup-norm" $|\cdot|_{\sup}$ of all values assumed by f:

$$\boxed{|f|_{\sup} = \sup_{x \in \mathrm{Max} A} |f(x)|}$$

First properties: $|f|_{sup}$ is only semi-norm! $|f|_{sup} = 0 \Leftrightarrow f$ nilpotent.

generally: $|f|_{sup} \leq |f|_\alpha$ any $\alpha : T_n \twoheadrightarrow A$. Choose $m \subset A$ maximal, let $n = \varphi^{-1} m \subseteq T_n$; $f \in A$ choose g preimage in T_n st. $|g| = |f|_\alpha$. Then $|f(m)| = |g(n)| \leq |g| = |f|_\alpha$

$$\begin{tikzcd} T_n/n \arrow[r, hook] & A/m \\ & K \arrow[ul, leftrightarrow] \arrow[u, hook] \end{tikzcd}$$

Proposition If $\varphi : B \to A$ is a K-morphism of affinoid K-algebras, $|\varphi(b)|_{sup} \leq |b|_{sup}$ $\forall b \in B$.

▷ If m is maximal in A, let $n := \varphi^{-1}(m) \subseteq B$. Then $K \hookrightarrow B/n \hookrightarrow A/m$, A/m fin. dim K-v.s.

$\Rightarrow$ $B/n \hookrightarrow A/m$ is finite, thus n is maximal. Then $\underset{\text{in } B/n}{|\bar{b}|} = \underset{\text{in } A/m}{|\overline{\varphi(b)}|}$, taking sup gives proposition /

Proposition On T_n we have that $|\cdot|_{sup} = |\cdot|$, where the latter is the Gauß norm on T_n

▷ max principle $|f| = \max\{|f(x)| ; x \in B^n(\bar{K})\}$

$x \in B^n(\bar{K})$ $m_x = \{h \in T_n / h(x) = 0\}$

$ev_x : T_n/m_x \hookrightarrow \bar{K}$ emb. Thus $f(x) = \bar{f}$

$x \mapsto m_x$ is surjection $B^n(\bar{K}) \twoheadrightarrow \mathrm{Max}\, T_n$

The maximum principle for affinoid K-algebras:

In order to prove this, need following lemma: if $T_d \hookrightarrow A$ is a finite monomorphism and A is torsion free, then for $m_x \in \mathrm{Max}\, T_d$ consider $m_{y_1}, \dots, m_{y_s}$ st. $m_{y_i} \cap T_d = m_x$. Then $\max_{i=1,\dots,s} |f(y_i)| = \max_{i=1,\dots,r} |a_i(x)|^{\frac{1}{i}}$ where a_i are the coefficients of the unique monic $p_f = t^r + a_1 t^{r-1} + \dots + a_r \in T_d[t]$ st $p_f(f) = 0$. In addition $|f|_{sup} = \max_{i=1,\dots,r} |a_i|_{sup}^{\frac{1}{i}}$

Theorem (maximum principle) For any affinoid K-algebra A, $f \in A$ there ex. $x \in \mathrm{Max}\, A$ s.t. $|f(x)| = |f|_{sup}$

▷ Proof. reduce to irreducible component of A: i.e. consider min. primes $\{p_1, \dots, p_s\}$ of A, then there ex. p_j st $|f_j|_{sup} = |f|_{sup}$

$A = T_n / p_1 \cdots p_s \longrightarrow T_n/p_j$ $\cancel{\alpha} \leftarrow \alpha$. I.e. wlog A is integral domain. Then apply Noether normalization to get a finite mono. $T_d \hookrightarrow A$. Now derive max. principle for A from max. principle for T_d: by the lemma, $f \in A$ satisfies some $f^r + a_1 f^{r-1} + \dots + a_r = 0$ over T_d and we have $\max_{j=1,\dots,s} |f(y_j)| = \max_{i=1,\dots,r} |a_i(x)|^{\frac{1}{i}}$ for any $x \in \mathrm{Max}\, T_d$ and $\mathrm{Max}\, A \ni \{y_1, \dots, y_s\} \longmapsto \{x\}$. As $a_i \in T_d$, apply max. principle to Tate algebra T_d: there ex. $x \in \mathrm{Max}\, T_d$ such that $|a_1(x)| \cdots |a_r(x)| = |(a_1 \cdots a_r)(x)| = |a_1 \cdots a_r| = |a_1| \cdots |a_r|$. This implies $|a_i(x)| \overset{(\text{all } i)}{=} |a_i|$ as "$\leq$" holds always and hence $\max_{j=1,\dots,s} |f(y_j)| = \max_{i=1,\dots,r} |a_i(x)|^{\frac{1}{i}} = \max_{i=1,\dots,r} |a_i|^{\frac{1}{i}} = |f|_{sup}$, so the sup is obtained by some element $y_j \in \mathrm{Max}\, A$.

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Power boundedness and topological nilpotency

Theorem. For $f \in A$ affinoid$/K$, $|.|_\alpha$ any residue norm on A, then for any $f \in A$, TFAE

(i) $|f|_{sup} \leq 1$

(ii) $\exists$ integral equation $f^r + a_1 f^{r-1} + \ldots + a_r = 0$, $a_i \in A$ s.t. $|a_i|_\alpha \leq 1$

(iii) the sequence $(|f|_\alpha^n)_{n\in\mathbb{N}}$ is bounded. say: f is power bounded wrto $|.|_\alpha$

▷ let $\alpha: T_n \twoheadrightarrow A$ be the epi defining $|.|_\alpha$. By N.n we have $T_d \to T_n \xrightarrow{\alpha} A$ for some $d \geq 0$ st

on T_d: $|.| = |.|_{sup}$

$T_d \hookrightarrow A$ is finite. Then any f satisfies $f^r + a_1 f^{r-1} + \ldots + a_r = 0$, $a_i \in T_n$ and $|a_i|_{sup} = |a_i| \leq 1$ by previous lemma

now recall that $T_d \hookrightarrow T_n$ is contractive wrto the Gauß norms, then $|\bar{a}_i|_\alpha \leq 1$ $\bar{a}_i$ images in A under α. $\Rightarrow$ (i) $\Rightarrow$ (ii) ✓

assume $\exists$ integral equation for f: write $A_0 := \{g \in A; |g|_\alpha \leq 1\}$, then (ii) means that f is integral over $A^\circ \Leftrightarrow A^\circ[f]$ is finite A°-module

and it follows that the sequence $(|f^n|_\alpha)$ must be bounded. (iii) $\Rightarrow$ (i) follows from the fact that $|f|_{sup}^n = |f^n|_{sup} \leq |f|_\alpha^n$ //

Corollary A affinoid, $f \in A$ and $|.|_\alpha$ a res. norm on A. TFAE

(i) $|f|_{sup} < 1$

(ii) $(|f^n|_\alpha)_n$ is a zero sequence. we call f topological nilpotent wrto $|.|_\alpha$

Remark notion of top. nilpotency is independent of the residue norm.

▷ Proof. follows if $|f|_{sup} = 0$. Thus assume $0 < |f|_{sup} < 1$. Then $\exists r$ st

$|f^r|_{sup} = |c| \in K^*$. $\Rightarrow |c^{-1} f^r|_{sup} = 1$, so $c^{-1} f^r$ is power-bounded wrto any $|.|_\alpha$. $\lim_{n\to\infty} |f^n|_\alpha = 0$

say $|c^{-n} f^{rn}|_\alpha \leq M$, $n \in \mathbb{N}$ and some $M \in \mathbb{R}$. $\Leftrightarrow |f^{rn}|_\alpha \leq c^n M$, so f^r is top. nilpotent

but then f^r is top. nilpotent, but then f is top. nilpotent. Conversely, assume $\lim_{n\to\infty} |f^n| = 0$, then $|f|_{sup}^n = |f^n|_{sup} \leq |f^n|_\alpha \xrightarrow{n\to\infty} 0$

$\Rightarrow |f|_{sup} < 1$ //

Lemma 19 Let A be affinoid K-algebra and let $f_1, \ldots, f_n \in A$.

(i) assume $\exists$ K morphism $\varphi: K\langle\zeta_1, \ldots, \zeta_n\rangle \to A$ such that $\varphi(\zeta_i) = f_i$ $(i = 1, \ldots, n)$. Then $|f_i|_{sup} \leq 1$ for all i

(ii) if $|f_i|_{sup} \leq 1$ for all i, then there ex. a unique K-morphism $\varphi: K\langle\zeta_1, \ldots, \zeta_n\rangle \to A$ st $\varphi(\zeta_i) = f_i$ (all i) and φ is continuous wrto the Gauß norm on $K\langle\zeta_1, \ldots, \zeta_n\rangle$ and any residue norm on A

(i) clear. (ii) define φ by above recipe. Now $|f_i|_{sup} \leq 1$ implies power-boundedness and thus well-defined and unique as a continuous morphism $\zeta_i \mapsto f_i$. Left to show, there ex. no other K-morphism $\varphi': K\langle\zeta_1, \ldots, \zeta_n\rangle \to A$ w/ $\zeta_i \mapsto f_i$. First reduce to case where

A is fin. dim. vector space$/K$. Claim: any K mor. $\varphi': K\langle\zeta_1, \ldots, \zeta_n\rangle \to A$ is continuous : have product top. on $A \xrightarrow{\sim} K^d$

induced by norm of the complete field K. Now view $T_n/\ker\varphi'$ as affinoid w/ can. residue norm, then $T/\ker\varphi' \hookrightarrow A$ is continuous, which can be reduced to the fact that linear forms on f.d. $V \to K$ are cont. for the product topology, thus φ' is continuous.

General: $T_n \overset{\varphi}{\underset{\varphi'}{\rightrightarrows}} A$. Then for $\mathfrak{m} \subseteq A$ maximal and some $r>0$ $A/\mathfrak{m}^r$ is a finite K v.s. $\Rightarrow$ maps $T_n \rightrightarrows A/\mathfrak{m}^r$ coincide (because are continuous)

Last claim: if $f \equiv 0 \mod \mathfrak{m}^r$ (all $\mathfrak{m}$ max and $r>0$) impl. $f=0$: Krull int-thm states $\bigcap_r \mathfrak{m}^r A_{\mathfrak{m}} = 0$ //

why is that interesting?

Proposition any morphism $\varphi: B \to A$ between affinoid K algebras A, B is continuous wrt any residue norm on A, B.

▷ **Proof:** choose $T_n \overset{\alpha}{\twoheadrightarrow} B \to A$. Then $T_n \to A$ is continuous, but then $B \to A$ must be continuous as well.

Preview

$\mathrm{Sp}\,A \subseteq \mathrm{Sp}\,T_n \qquad A = T_n/\mathfrak{a}$

$V(\mathfrak{a}) = \mathrm{Sp}\,A = X$

$\mathbb{B}^n(\bar{K})$ from $\bar{K}^n$

$U \longrightarrow X$

$\mathrm{Sp}\,A' \longrightarrow \mathrm{Sp}\,A$

$A \longrightarrow A'$

$\mathfrak{p}' \cap A \longleftarrow \mathfrak{p}'$

$\mathfrak{p}' \cap A = \mathfrak{p}$

$\mathfrak{p}' \longmapsto \mathfrak{p}A \neq \mathfrak{p}'$

$Y' = V(\mathfrak{p}') \rightsquigarrow V(\mathfrak{p}')$ closure in X $= V(\mathfrak{p}' \cap A) \cap U$

$\boxed{\text{Ex. 22}}$