Rigid-analytic spaces II

22 June 2021 10:59

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Non-archimedean geometry study group

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Functoriality of spectrum

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In algebraic geometry, many things are dictated by functoriality. We wish for this to hold in rigid geometry too.

$A \cong K\langle T_1,\dots,T_n\rangle / I$

Proposition

Let $\sigma : A \to B$ be a morphism of affinoid K-algebras. For any maximal $\mathfrak{m} \in \operatorname{Sp} B$ we have $\sigma^*(\mathfrak{m}) := \sigma^{-1}(\mathfrak{m}) \in \operatorname{Sp} A$.

Proof: $K \to A/\sigma^{-1}(m) \longrightarrow B/m$ fin. ext. of K

a field too $\Rightarrow \sigma^{-1}(m)$ max. $\square$

Functoriality of spectrum

In algebraic geometry, many things are dictated by functoriality. We wish for this to hold in rigid geometry too.

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Definition

A *morphism* of affinoid spaces is any map of the form $\sigma^* : \operatorname{Sp} B \to \operatorname{Sp} A$.

Weierstrass domains

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$\{\sum a_n f^n \mid |a_n| \to 0\}$

$K\langle T\rangle$ - alg. of fns on closed disk - Banach algebra

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Proposition

Let $\sigma : A \to B$. If the image of $\sigma^* : \operatorname{Sp} B \to \operatorname{Sp} A$ is contained in $X(f)$, then σ uniquely extends through $A\langle f\rangle$.

Proof: $\sigma^*(\operatorname{Sp} B) \subseteq X(f) \iff |\sigma(f)(y)| \leq 1$ for $y \in \operatorname{Sp} B$

$\iff \sigma(f)$ is power-bounded

$\Rightarrow \sum a_n \sigma(f)^n$ w/ $|a_n| \to 0$ converges

$\rightsquigarrow A\langle T\rangle \to B$, $T \mapsto \sigma(f)$, vanishes on $T - f$

$\rightsquigarrow A\langle f\rangle \to B$ $\square$

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More generally, for a tuple $f_1, \ldots, f_n \in A$ we can consider

$$X(f) = X(f_1, \ldots, f_n) = \{x \in X \mid \forall i : |f_i(x)| \leq 1\}.$$

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Weierstrass domains form a basis of the *canonical topology* on $\operatorname{Sp} A$. It agrees with the one coming from $\overline{K}^m$.

Laurent and rational domains

In similar vein we have *Laurent domains*: for $f_i, g_j \in A$ we have

$$X(f, g^{-1}) = \{x \in X \mid \forall i : |f_i(x)| \le 1, \quad \forall j : |g_j(x)| \ge 1\}.$$

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Its algebra of functions is $A\langle f, g^{-1}\rangle = A\langle T, U\rangle/(T - f, g\overset{U}{T} - 1)$.

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We also have rational domains: for $f_i \in A$ and $g \in A$ *which have no common zeros* we let

$c \in K$

$g = c \cdot f \quad |c| < 1$

$$X\left(\frac{f}{g}\right) = \{x \in X \mid \forall i : |f_i(x)| \le |g(x)|\}.$$

$|f| \le |c| \cdot |f|$

Laurent and rational domains

$\mathrm{Sp}\, K\langle T\rangle \ni m_a$

$\updownarrow$

$a \in \overline{B}_1(\overline{K})$

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$A\langle T_1, \dots, T_n, U_1, \dots, U_m\rangle/(T_i - f_i, g_j U_j - 1)$

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$$X\left(\frac{f}{g}\right) = \{x \in X \mid \forall i : |f_i(x)| \le |g(x)|\}.$$

$A\langle \frac{1}{g}\rangle$ – universal A-alg. in which g has powerbounded inverse

The associated algebra is $A\langle \frac{f}{g}\rangle = A\langle T\rangle/(gT - f)$. Note that it contains $\frac{1}{g}$.

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- The map $\operatorname{Sp} A_U \to \operatorname{Sp} A$ has image contained in U,
- Any morphism $A \to B$ such that $\operatorname{Sp} B$ is mapped into $U \subseteq \operatorname{Sp} A$ factors uniquely through A_U.

eg. $U = X(f)$, $A_U = A\langle f \rangle$

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Proposition

- $\operatorname{Sp} A_U \to \operatorname{Sp} A$ *induces a homeomorphism onto* U. $\rightsquigarrow$ identify U w/ $\operatorname{Sp} A_U$

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Proposition

- $\operatorname{Sp} A_U \to \operatorname{Sp} A$ *induces a homeomorphism onto* U.
- *Weierstrass (rational) domain in a Weierstrass (rational) domain is a Weierstrass (rational) domain.*

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Proposition

- *$\operatorname{Sp} A_U \to \operatorname{Sp} A$ induces a homeomorphism onto U.*
- *Weierstrass (rational) domain in a Weierstrass (rational) domain is a Weierstrass (rational) domain.*
- *Every rational domain is a Weierstrass domain in a Laurent domain.*

Further properties

Proposition (Continued)

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- *Affinoid domain in an affinoid domain is an affinoid domain.*
- *Intersection of two affinoid domains is affinoid.*

$\mathcal{U} \rightsquigarrow A_{\mathcal{U}}$

$V \rightsquigarrow A_V$

$\mathcal{U} \cap V \rightsquigarrow A_{\mathcal{U}} \hat{\otimes} A_V$

$K\langle T\rangle \hat{\otimes} K\langle \mathcal{U}\rangle \cong K\langle T, \mathcal{U}\rangle$

Further properties

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- *Intersection of two affinoid domains is affinoid.*
- *Disjoint union of two affinoid domains is affinoid.*

$A_{\mathcal{U}} \oplus A_V$

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- *Intersection of two affinoid domains is affinoid.*
- *Disjoint union of two affinoid domains is affinoid.*
- *Every affinoid domain is open.*
- *(Gerritzen-Grauert) Every affinoid domain is a finite union of rational domains.*

Structure sheaf, first attempt

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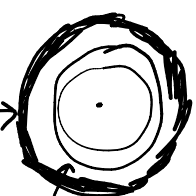
$A = K\langle T\rangle$, $\quad \operatorname{Sp} A =$ unit disk



$1 \in K\langle T, T^{-1}\rangle \qquad \operatorname{Sp} K\langle T, T^{-1}\rangle$

$0 \in K\langle c^{-1}T\rangle \qquad \operatorname{Sp} K\langle c^{-1}T\rangle$, $c \in K$, $|c| < 1$

$\rightsquigarrow$ disk of rad. $|c|$.

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It seems like this space has too many open ~~sets.~~ coverings

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- a collection of *admissible open subsets*,
- for each admissible U, a collection of *admissible coverings* of U by other admissibles,

subject to the following conditions:

- Intersection of two admissible opens is admissible open,
- $\{U\}$ is a cover of U,
- If $\{U_i\}_i$ is a cover of U and $U' \subseteqq U$, then $\{U_i \cap U'\}$ is a adm. cover of U',
- If $\{U_i\}_i$ is a cover of U and $\{V_{ij}\}_j$ is a cover of U_i, then $\{V_{ij}\}_{i,j}$ is a cover of U.

Weak G-topology

On any affinoid space $\operatorname{Sp} A$ we have a *weak G-topology*: admissible opens are the affinoid domains, and admissible covers are the *finite covers*.

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The assignment $O_X(U) = A_U$ defines a sheaf in the weak G-topology. More precisely, for $U = U_1 \cup \cdots \cup U_n$, the Cech complex

$$0 \to A_U \to \prod A_{U_i} \to \prod A_{U_i \cap U_j} \to \prod A_{U_i \cap U_j \cap U_k} \to \cdots$$

is exact.

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Therefore the structure sheaf is an *acyclic* sheaf.

Proof of Tate acyclicity

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$f_0, \dots, f_n \in A, \quad X = \bigcup_i X\left(\frac{f_0}{f_i}, \dots, \frac{f_n}{f_i}\right)$

$(f_0, \dots, f_n) = A$

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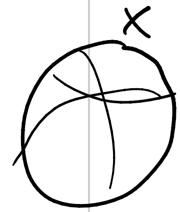
- Pass to elements of a Laurent covering $\to$ may assume the rational coverings are generated by units.



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- Rational covering generated by units can be refined by a Laurent covering $\to$ can reduce to "standard" Laurent coverings.

$f_1, \dots, f_n$, $X(f_1^{\pm 1}, f_2^{\pm 1}, \dots, f_n^{\pm 1})$

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- Inductive reasoning $\to$ enough to show the result for $X = X(f) \cup X(f^{-1})$.

$$0 \to A \to A\langle f\rangle \oplus A\langle f^{-1}\rangle \to A\langle f, f^{-1}\rangle \to 0$$
$$(x, y) \mapsto x - y$$

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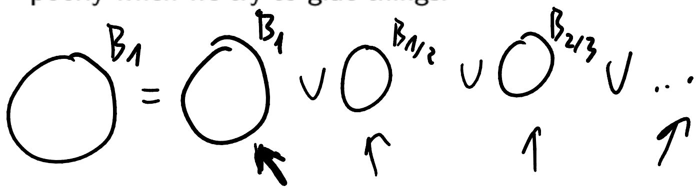
- Inductive reasoning $\to$ enough to show the result for $X = X(f) \cup X(f^{-1})$.
- This last case is done by direct calculation.

Weak topology is too weak

Weak G-topology has some properties which make it behave poorly when we try to glue things.



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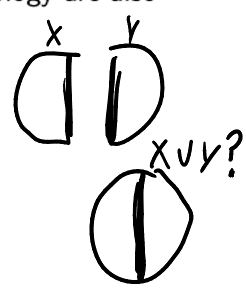

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"admissibility is G-local"

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Theorem

For any G-topology T on a set X there is a canonical refinement T' which satisfies (G_1) and (G_2).
Furthermore, T' is slightly finer by T, which ~~means~~ implies sheaves on T extend uniquely to T', as do morphisms of sheaves.

$\mathrm{Shv}(X,T) \cong \mathrm{Shv}(X,T')$ "as topoi/toposes"

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finite "as far as affinoid spaces can see"

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- A cover $\{U_i\}$ of admissible U is admissible if for all $\varphi : Z \to X$ with $\varphi(Z) \subseteq U$, the cover $\{\varphi^{-1}(U_i)\}$ of Z has a finite refinement by affinoid domains.

Open unit disk: $B_1^0 = \bigcup_{\substack{r<1 \\ r \in |k^\times|}} \overline{B}_r \subseteq \overline{B}_1$

$\varphi : Z \to \overline{B}_1$, $\varphi(Z) \subseteq B_1^0$ $\rightsquigarrow$ function f on Z s.t. $|f(z)| < 1$ for all $z \in Z$

$\Rightarrow \exists r < 1$ s.t. $|f(z)| \leq r$ $\forall z \in Z$

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$U = \{f \neq 0\}$

$U = \bigcup_{r>0} \{|f| \geq r\}$

Proposition

- *The strong G-topology satisfies $(G_0), (G_1)$ and (G_2).*
- *All finite unions of affinoid domains are admissible open, and such unions are admissible covers.*
- *All Zariski open subsets are admissible open, and their arbitrary unions are admissible covers.*

Rigid-analytic spaces

Rigid-analytic spaces

Definition

A *(locally) G-ringed space* is a set X equipped with a G-topology and a sheaf of rings O_X (such that all stalks are local rings.)

e.g. Sp A with either weak or strong G-top.

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- X satisfies $(G_0), (G_1), (G_2)$.

Rigid-analytic spaces

Definition

A *(locally) G-ringed space* is a set X equipped with a G-topology and a sheaf of rings O_X (such that all stalks are local rings.)

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A *rigid-analytic space* over K is a locally G-ringed space (X, O_X) such that

- there is an admissible covering $\{X_i\}$ such that each $(X_i, O_X|_{X_i})$ is isomorphic to an affinoid space,
- X satisfies $(G_0), (G_1), (G_2)$.

Proposition (Gluing rigid spaces)

Suppose we are given rigid spaces X_i, admissible opens X_{ij} $\subset X_i$ and isomorphisms $\varphi_{ij} : X_{ij} \to X_{ji}$ satisfying suitable cocycle conditions. Then they can be uniquely glued to one space X of which $\{X_i\}$ is an admissible covering.

Wojtek Wawrów Rigid-analytic spaces II

Example: affine space

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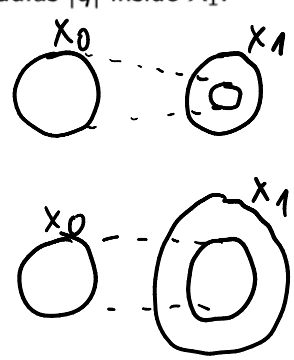

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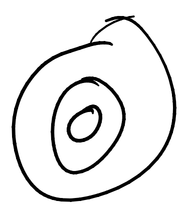
$$X_0 \xrightarrow{\varphi^*} X_1 \xrightarrow{\varphi^*} X_2 \xrightarrow{\varphi^*} \dots$$



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Gluing these together we get the *rigid-analytic affine line* $\mathbb{A}^{1,rig}$.

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$\overline{K}/\mathrm{gal}$

Its underlying set of points is the same as $\operatorname{Spec}_{an} K[T]$, but the ring of functions is larger: it contains all globally convergent power series.

Coherent sheaves (if time permits)

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Let M be an A-module. Then $\widetilde{M} : U \mapsto A_U \otimes_A M$ defines an acyclic sheaf of O_X-modules on $\operatorname{Sp} A$.

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For a rigid space X and a sheaf F of O_X-modules, the following conditions are equivalent:

- There is an admissible cover by affinoid subspaces U_i such that $F|_{U_i} \cong \widetilde{M_i}$ for some f.g. $O_X(U_i)$-module M_i,
- Above holds for *all* admissible covers by affinoid subspaces,
- F is locally of finite presentation on X. $\mathcal{O}_X^n \to \mathcal{O}_X^m \to F \to 0$

If this condition is satisfied, F is called *coherent*.

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Corollary (Kiehl's Theorem)

Every coherent O_X-module on $\operatorname{Sp} A$ *is of the form* $\widetilde{M}$.