Berkovich Spaces I

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Let X be an algebraic variety over k, an algebraically closed, complete, non-archimedean field with non-trivial absolute value. We can try to assign $X(k)$ the weakest topology such that for every Zariski open $U \subset X$ and every regular function $f \in \mathcal{O}_X(U)$, the function $U(k) \to \mathbb{R} : P \mapsto \left|f(P)\right|$ is continuous.

Tate: use a Grothendieck topology. But this implies the existence of non-zero abelian sheaves with all stalks zero: we need more points.

Berkovich: points with values in fields with rank 1 valuations. Spaces become Hausdorff and locally compact.

Huber considered arbitrary valuations, giving adic spaces. These are not Hausdorff in general.

All rings are commutative with identity.

A *seminorm* on a ring $\mathscr{A}$ is a function $|\ |: \mathscr{A} \to \mathbb{R}_{\geqslant 0}$ such that $|0| = 0, |1| = 1$, $|f+g| \leqslant |f| + |g|$ and $|fg| \leqslant |f| \cdot |g|$ for all $f, g \in \mathscr{A}$.

If $|fg| = |f| \cdot |g|$, we say that $|\ |$ is multiplicative.

Each seminorm $|\ |$ determines a topology, which is Hausdorff if and only if $|\ |$ is a norm, i.e. $|f| = 0 \Rightarrow f = 0$. One can construct a completion of $\mathscr{A}$ with respect to any seminorm.

We say that two seminorms $|\ |$ and $|\ |'$ are *equivalent* if there exist $C, C' > 0$ such that $C|f| \leqslant |f|' \leqslant C'|f|$ for all $f \in \mathscr{A}$. Equivalent seminorms define the same topology. If $\mathbf{a}$ is an ideal of $\mathscr{A}$ we can define the *residue seminorm* on $\mathscr{A}/\mathbf{a}$ by $|f| = \inf\{|g| \mid g \text{ projects to } f\}$.

The residue seminorm is a norm if and only if $\mathbf{a}$ is closed in $\mathscr{A}$.

Lemma 1.

If $\mathscr{A}$ is a normed ring and $f \in \mathscr{A}$ is invertible, then for all $m, n \geqslant 0$ we have $\|f^n\|^{-m} \leqslant \left\|f^{-m}\right\|^n$.

Lemma 2.

If $\sum_i a_i < \infty$ and the a_i are all positive real numbers, then $\sum_i a_i^n < \infty$ for all $n \geqslant 1$.

Let $\phi : \mathscr{A} \to \mathscr{B}$ be a homomorphism of seminormed rings. We say that ϕ is *bounded* if there is a constant $C > 0$ such that $\left|\phi(f)\right| \leqslant C|f|$ for all $f \in \mathscr{A}$.

A *Banach ring* is a normed ring that is complete with respect to its norm.

Examples. (i) The *trivial norm* $|\ |_0$ with $|f|_0 = 1$ for all $f \neq 0$ makes any ring into a Banach ring.

(ii) If $\mathscr{A}$ is Banach and **a** is a closed ideal, then $\mathscr{A}/\mathbf{a}$ is complete with respect to the residue norm. The set of invertible element in $\mathscr{A}$ is open and an element $x \in \mathscr{A}$ is not invertible if and only if it lies in some maximal ideal of $\mathscr{A}$. It follows that every maximal ideal is closed.

(iii) $\mathbb{Z}$ is a Banach ring with respect to the usual absolute value $|\ |_\infty$.

(iv) For a Banach ring $\mathscr{A}$ and a positive number r, let $\mathscr{A}\langle\langle r^{-1}T\rangle\rangle$ denote the set of power series $f = \sum_{i=0}^{\infty} a_i T^i$ such that $\sum_{i=0}^{\infty} \|a_i\|\, r^i < \infty$. Then $\mathscr{A}\langle\langle r^{-1}T\rangle\rangle$ is Banach with respect to the norm $\|f\| = \sum_{i=0}^{\infty} \|a_i\|\, r^i$. Note that an element $1 - aT$, with $a \in \mathscr{A}$, is invertible in $\mathscr{A}\langle\langle r^{-1}T\rangle\rangle$ if and only if $\sum_{i=0}^{\infty} \|a^i\|\, r^i < \infty$.

Let $\mathscr{A}$ be a Banach ring. The *spectrum* $\mathscr{M}(A)$ is the set of all bounded multiplicative seminorms on $\mathscr{A}$ equipped with the weakest topology such that the evaluation functions $|\ | \mapsto |f|$, for $f \in \mathscr{A}$, are continuous. Our first goal is the following theorem.

Theorem.

The spectrum $\mathscr{M}(\mathscr{A})$ is a nonempty compact Hausdorff space.

Let $\phi : \mathscr{A} \to \mathscr{B}$ be a bounded homomorphism of Banach rings. It induces a continuous map $\phi^* : \mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$. Assume that the set of elements of the form $\phi(f)/\phi(g)$ for $f, g \in \mathscr{A}$ and $\phi(g)$ invertible in $\mathscr{B}$ is dense in $\mathscr{B}$. Then by continuity, every seminorm of $\mathscr{B}$ is determined by its values on this set, i.e. ϕ^* is injective.

We are ready to prove that $\mathscr{M}(\mathscr{A})$ is nonempty.
By the previous observation it is enough to show that $\mathscr{M}(\mathscr{A}/\mathbf{m})$ is nonempty, where $\mathbf{m}$ is a maximal ideal. So assume that $\mathscr{A}$ is a field. Let S be the set of nonzero bounded seminorms on $\mathscr{A}$. S is nonempty because it contains the norm of $\mathscr{A}$. We put a partial order on S: $|\ | \leqslant |\ |'$ if $|f| \leqslant |f|'$ for all $f \in \mathscr{A}$. Let $|\ |$ be a minimal element in S. We can replace $\mathscr{A}$ by its completion with respect to $|\ |$ and thus assume that $|\ |$ is the norm of $\mathscr{A}$. Now we need to show that $|\ |$ is multiplicative.
First we prove that $|f^n| = |f|^n$ for all $f \in \mathscr{A}$. If not, there exists an f such that $|f^n| < |f|^n$ for some n. We claim that $f - T$ is not invertible in the Banach ring $\mathscr{A}\langle\langle r^{-1}T\rangle\rangle$ where $r = |f^n|^{1/n}$. The inverse of $f - T$ in $\mathscr{A}[[T]]$ is $f^{-1}(1 - f^{-1}T)^{-1}$. Therefore it is enough to show that $\sum_{i=0}^{\infty} \left|f^{-i}\right| r^i$ does not converge. Applying Lemma 1, we have

$$\infty = \sum_{i=0}^{\infty} 1 = \sum_{i=0}^{\infty} r^{-in} r^{in} = \sum_{i=0}^{\infty} |f^n|^{-i} r^{in} \leqslant \sum_{i=0}^{\infty} \left|f^{-i}\right|^n r^{in}.$$

By Lemma 2, this implies that $f - T$ is not invertible in $\mathscr{A}\langle\langle r^{-1}T\rangle\rangle$.

Consider the homomorphism $\phi : \mathscr{A} \to \mathscr{A}\langle\langle r^{-1}T\rangle\rangle/(f - T)$. Since $\mathscr{A}$ is a field, this is injective, and $\left\|\phi(f)\right\| = \|T\| = r = |f^n|^{1/n} < |f|$. Pulling back the residue norm on $\mathscr{A}\langle\langle r^{-1}T\rangle\rangle/(f - T)$ to $\mathscr{A}$ we get a seminorm $|\ |$ satisfying $|f|' < |f|$. This is impossible as $|\ |$ is a minimal seminorm.
Similarly we can prove that for nonzero $f \in \mathscr{A}$ we have $\left|f^{-1}\right| = |f|^{-1}$. Together with submultiplicativity this gives

$$|fg|^{-1} = \left|f^{-1}g^{-1}\right| \leqslant \left|f^{-1}\right|\left|g^{-1}\right| = |f|^{-1}|g|^{-1} \leqslant |fg|^{-1}.$$

Thus $|\ |$ is multiplicative and $\mathscr{M}(\mathscr{A})$ is nonempty.

Let x' and x'' be distinct points of $\mathscr{M}(\mathscr{A})$. Without loss of generality, there exists $f \in \mathscr{A}$ with $|f|_{x'} < |f|_{x''}$. Pick a real number r with $|f|_{x'} < r < |f|_{x''}$. Then $U' = \{x \in \mathscr{M}(\mathscr{A}) \mid |f|_{x'} < r\}$ and $U'' = \{x \in \mathscr{M}(\mathscr{A}) \mid |f|_x > r\}$ are disjoint neighbourhoods of x' and x'' respectively. Hence $\mathscr{M}(\mathscr{A})$ is Hausdorff.

Let $|\ |_x$ be an element of $\mathscr{M}(\mathscr{A})$. The kernel $\mathfrak{p}_x$ is a closed prime ideal of $\mathscr{A}$. The value $|f|$ depends only on the residue class of f in $\mathscr{A}/\mathfrak{p}_x$. The resulting valuation on $\mathscr{A}/\mathfrak{p}_x$ extends to a valuation on its fraction field $K(x)$. The completion of $K(x)$ with respect to $|\ |_x$ is a complete valued field denoted by $\mathscr{H}(x)$. The image of $f \in \mathscr{A}$ in $\mathscr{H}(x)$ is denoted by $f(x)$. The homomorphism

$$\mathscr{A} \longrightarrow \prod_{x \in \mathscr{M}(\mathscr{A})} \mathscr{H}(x)$$

which sends f to $\hat{f} = (f(x))_{x \in \mathscr{M}(\mathscr{A})}$ is called the *Gel'fand transform*.
Set $\mathscr{B} = \prod_{x \in \mathscr{M}(\mathscr{A})} \mathscr{H}(x)$. The induced map $\mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ is surjective. Compactness of $\mathscr{M}(\mathscr{A})$ follows from the following result.

Theorem.

Let $\{K_i\}_{i \in I}$ be a family of valuation fields. Then the spectrum $\mathscr{M}(\mathscr{B})$ of $\mathscr{B} = \prod_{i \in I} K_i$ is homeomorphic to the Stone-Čech compactification of the discrete set I.

This is proved using filters.

There is a natural continuous map $\mathscr{M}(\mathscr{A}) \to \mathrm{Spec}(\mathscr{A}) : x \mapsto \mathfrak{p}_x$, sending a seminorm to its kernel.
Let's look at $\mathscr{M}(\mathbb{Z})$ more closely. By Ostrowski's theorem, any multiplicative seminorm on $\mathbb{Z}$ is one of the following:

(1) A seminorm $| \ |_{\infty,\varepsilon}$, where $\varepsilon \in (0, 1]$:

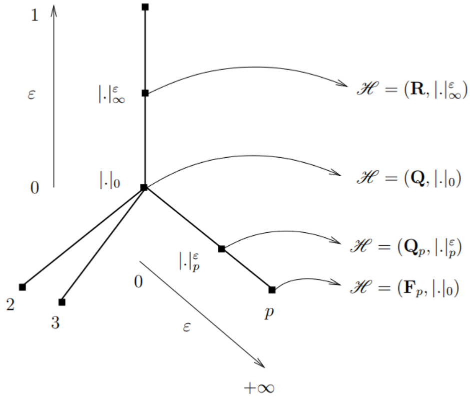
$$|n|_{\infty,\varepsilon} = |n|_{\infty}^{\varepsilon}$$

(2) The trivial seminorm $| \ |_0$

(3) A p-adic seminorm $| \ |_{p,\varepsilon}$, where p is a prime and $|n|_{p,\varepsilon} = \varepsilon^{-v_p(n)}$ for $\varepsilon \in (0, \infty)$

(4) A p-trivial seminorm $| \ |_{p,0}$ where p is a prime and

$$|n|_{p,\infty} = \begin{cases} 0, & \text{if } p \mid n. \\ 1, & \text{otherwise.} \end{cases}$$



The natural map $\mathscr{M}(\mathscr{A}) \to \mathrm{Spec}(\mathbb{Z})$ sends the lower endpoints of the prime number intervals to their respective prime ideals and all other points to the generic point of $\mathrm{Spec}(\mathbb{Z})$.

Fix $n \in \mathbb{Z}$. The preimage of $\{a < |n|_x < b\} \subset \mathbb{R}_{\geqslant 0}$ is open in $\mathscr{M}(\mathbb{Z})$. The open neighbourhoods of $|\ \ |_0$ in $\mathscr{M}(\mathbb{Z})$ are those subsets which contain all branches except finitely many, and contain a Euclidean neighbourhood of $|\ \ |_0$ in the remaining ones.

Let k be a complete, algebraically closed, non-archimedean field.
The *Berkovich affine line* $\mathbb{A}_k^{1,\mathrm{an}}$ is the set of multiplicative seminorms on the polynomial ring $k[T]$ which restrict to the valuation on k. This set is given the weakest topology such that all real-valued functions of the form $|\ | \mapsto |f|$, for $f \in k[T]$, are continuous.

Let $k\langle r^{-1}T\rangle = \{f = \sum_{i=0}^{\infty} \mid \sup |a_i|\, r^i < \infty\}$ be the Tate algebra, which is a Banach ring with respect to the norm $\|f\| = \sup |a_i|\, r^i$. Set $X = \mathscr{M}(k\langle r^{-1}T\rangle)$. For $a \in k$ and $\rho \in \mathbb{R}$ with $|a| \leqslant r$ and $0 < \rho \leqslant r$, define $D(a, \rho) = \{x \in X \mid |x - a| \leqslant \rho\}$. We have $D(0, r) = X$.
Furthermore $\mathbb{A}_k^{1,\mathrm{an}} \cong \bigcup_{r>0} D(0, r)$, so $\mathbb{A}_k^{1,\mathrm{an}}$ is locally compact.
Proof. We define continuous maps in each direction. The inclusion $k[T] \to k\langle r^{-1}T\rangle$ induce maps $\iota_r : D(0, r) \to \mathbb{A}_k^{1,\mathrm{an}}$. These maps are compatible, so they induce

$$\iota : \varinjlim D(0, r) = \bigcup_{r>0} D(0, r) \to \mathbb{A}_k^{1,\mathrm{an}}.$$

Suppose $x \in \mathbb{A}_k^{1,\mathrm{an}}$, let $r = |T|_x$. Define $\psi(x) \in D(0, r)$ by

$$|f|_{\psi(x)} = \lim_{n\to\infty} \left| \sum_{i=0}^{n} a_i T^i \right|_x$$

for $f \in k\langle r^{-1}T\rangle$. The resulting map $\psi : \mathbb{A}_k^{1,\mathrm{an}} \to \varinjlim D(0, r)$ is inverse to ι.

Our next goal is to characterise the points of $\mathbb{A}_k^{1,\mathrm{an}}$.
Each element a of k gives rise to a point of $\mathbb{A}_k^{1,\mathrm{an}}$ via the seminorm

$$| \ |_a : f \in k[T] \mapsto |f(a)| \in \mathbb{R}_{\geqslant 0}.$$

Conversely a can be recovered from $| \ |_a$ since $\ker(| \ |_a) = (T - a)$. This gives an injection $k \hookrightarrow \mathbb{A}_k^{1,\mathrm{an}}$. We can identify a point a of k with a generalised disk $D(a, 0)$ of radius 0. We call these type 1 points.
Any closed disk $D(a, r)$ with $r > 0$ defines a multiplicative seminorm given by $| \ |_D : k[T] \to \mathbb{R}_{\geqslant 0} : f \mapsto \sup_{x \in D} |f(x)|$. If r lies in the value group $|k^\times|$, this is a type 2 point. If $r \notin |k|$, it is a type 3 point.
In general, there is one more type of point: let $D_1 \supset D_2 \supset D_3 \supset \dots$ be a sequence of nested closed disks of k. If $\bigcup D_i = \varnothing$, then $\lim_{n\to\infty} |f|_{D_n}$ defines a new point of $\mathbb{A}_k^{1,\mathrm{an}}$.
Such points, called type 4 points, only occur when k is not spherically complete.
We say that k is spherically complete if every nested sequence of closed disk has nonempty intersection.

Theorem.

$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ is not spherically complete.

Proof. Let r_n be a sequence of real numbers converging to $r > 0$ from above. Set $D_0 = D(0, r_0)$. We can find disjoint disks D_1 and D_1' both of radius r_1 contained in D_0. Continuing in this fashion, we get $2^{\aleph_0}$ nested sequences of closed disks. Let $\mathscr{D}_\gamma$ be one such sequence and set $D_\gamma = \bigcap_{D \in \mathscr{D}_\gamma} D$.
Assume D_γ is nonempty, let $x \in D_\gamma$. Since x lies in D for all $D \in \mathscr{D}_\gamma$, it is the centre of each disk. So D_γ contains $D(x, r)$. Now if $|x - y| = r' > r$, then there is some $D \in \mathscr{D}_\gamma$ with radius less than r'. So $y \notin D$ and thus $y \notin D_\gamma$. Thus D_γ is either empty of a disk of radius r. In particular, each D_γ is open.
But $\mathbb{C}_p$ is separable ($\overline{\mathbb{Q}}$ is dense in it), so each nonempty D_γ must contain a point of $\overline{\mathbb{Q}}$. But the D_γ are disjoint, so all but countably many of them must be empty.

Remark. Every local field is sperically complete. Every valued field has a spherical completion.

Theorem.

Every point of $\mathbb{A}_k^{1,\mathrm{an}}$ can be realised as $\lim_{n\to\infty} |\ |_{D_n}$ for some nested sequence $D_1 \supset D_2 \supset \dots$ of closed disks in k.

Proof. Seminorms are multiplicative and k is algebraically closed, so we only need to consider seminorms of linear polynomials in $k[T]$. Fix a point $x \in \mathbb{A}_k^{1,\mathrm{an}}$. Consider the family $\mathcal{F}$ of closed disks

$$\mathcal{F} = \{D(a, |T-a|_x) \mid a \in k\}.$$

Let $a, b \in k$. Without loss of generality, $|T-a|_x \geqslant |T-b|_x$. We have $|a-b| = |a-b|_x \leqslant \max\{|T-a|_x, |T-b|_x\}$ with equality if $|T-a|_x < |T-b|_x$. Thus $b \in D(a, |T-a|_x)$ and so $D(b, |T-b|_x) \subset D(a, |T-a|_x)$. We see that $\mathcal{F}$ is totally ordered.
Let $r = \inf_{a\in k} |T-a|_x$. Choose a sequence of points a_n in k such that r_n decreases to r, where $r_n = |T-a_n|_x$. Set $D_n = D(a_n, r_n)$. This is a nested sequence of closed disks.
Now one can show that for all $a \in k$, $|T-a|_x = \lim_{n\to\infty} |T-a|_{D_n}$.

Summing up: let $| \ |_x \in \mathbb{A}_k^{1,\mathrm{an}}$, let $D_n = D(a_n, r_n)$ be the corresponding nested sequence of closed disks, and let $r = \lim r_n$.

Type 1: $r = 0$ and $\bigcap D_n = a$, where completeness of k ensures that the limit is nonempty.

Type 2: $\bigcap D_n = D(a, r)$ with $r \in \left|k^\times\right|$

Type 3: $\bigcap D_n = D(a, r)$ with $r \notin |k|$.

Type 4: $\bigcap D_n = \varnothing$.

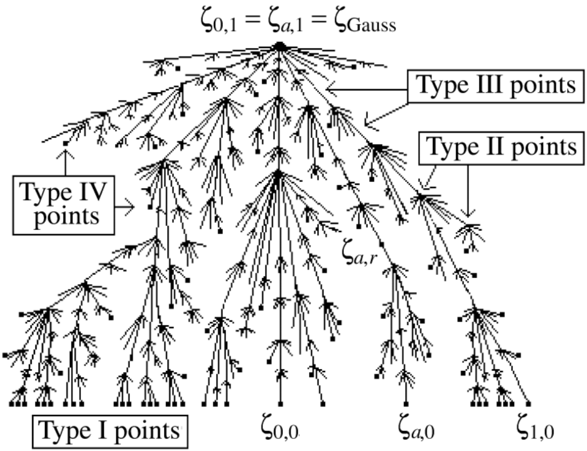


Figure: The Berkovich unit disk $D(0,1)$.

$\mathbb{A}_k^{1,\mathrm{an}}$ is uniquely path-connected.

Local description of the points of $\mathbb{A}_k^{1,\mathrm{an}}$:

For $x \in \mathbb{A}_k^{1,\mathrm{an}}$, define $T_x = \{\text{connected components of } \mathbb{A}_k^{1,\mathrm{an}} - \{x\}\}$.

If x is Type 1, then $\#T_x = 1$.

If x is Type 2, then $\#T_x$ is infinite: there are infinitely many branches joining at x.

If x is Type 3, there is no branching: $\#T_x = 2$.

If x is Type 4, $\#T_x = 1$: x is a dead end.

Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$. Recall that $K(x)$ is defined as the fraction field of $k[T]/\ker(|\ \ |_x)$ and $\mathscr{H}(x)$ is the completion of $K(x)$ with respect to $|\ \ |_x$. We can describe the points of $\mathbb{A}_k^{1,\mathrm{an}}$ in terms of the value group and the residue field $\widetilde{\mathscr{H}(x)}$ of $\mathscr{H}(x)$, where $\widetilde{\mathscr{H}(x)} = \mathcal{O}_x/\mathfrak{m}_x$ with $\mathcal{O}_x = \{f \in K(x) \mid |f|_x \leqslant 1\}$ and $\mathfrak{m}_x = \{f \in K(x) \mid |f|_x < 1\}$.

Type	Value group $\left\|\mathscr{H}(x)^\times\right\|$	Residue field $\widetilde{\mathscr{H}(x)}$
1	$\left\|k^\times\right\|$	$\tilde{k}$
2	$\left\|k^\times\right\|$	$\tilde{k}(t)$
3	$\langle \left\|k^\times\right\|, r\rangle$	$\tilde{k}$
4	$\left\|k^\times\right\|$	$\tilde{k}$

Here, $r \in \mathbb{R}\backslash\left|k^\times\right|$. Moreover, x is of type 1 if and only if $\mathscr{H}(x) \cong k$.

Proof. Type 1: x is given by $|f|_x = |f(a)|$. $\ker |\ |_x = (T - a)$, so $K(x) = \mathscr{H}(x) = k$.
Type 2: x corresponds to a disk $D(a, r)$ with $r \in |k^\times|$. No polynomial vanishes identically on $D(a, r)$, so $K(x) \cong k(T)$. For any $f \in k[T]$ there exists a $p \in D(a, r)$ such that $|f|_x = |f(p)|$ by the non-archimedean maximum principle. Therefore $|K(x)^\times| = |\mathscr{H}(x)^\times| = |k^\times|$. Take $c \in k^\times$ with $|c| = r$. Let t be the reduction of $(T - a)/c$ in the residue field $\widetilde{H(x)}$. One can show that t is transcendental over $\tilde{k}$ and $\widetilde{H(x)} = \widetilde{K(x)} = \tilde{k}(t)$.
Type 3: Suppose x corresponds to $D(a, r)$ with $r \notin |k^\times|$. We have $|T - a|_x = r$, and so $|\mathscr{H}(x)^\times| = \langle |k^\times|, r\rangle$. To prove that the residue field is $\tilde{k}$, write $f \in k(T)$ as a quotient of polynomials $f = g/h$,

$$g(T) = \sum b_i (T - a)^i, h(T) = \sum c_j (T - a)^j.$$

Notice that since $r \notin |k^\times|$, the strict ultrametric inequality applies to the terms of g and h: there are indices i_0 and j_0 such that $|g|_x = |b_{i_0}| r^{i_0}, |h|_x = |c_{j_0}| r^{j_0}$. If $|f|_x = 1$, we must have $i_0 = j_0$ and

$$f \equiv b_{i_0}/c_{i_0} \mod \mathfrak{m}_x.$$

So every $f \in \mathcal{O}_x$ has constant reduction, i.e. $\widetilde{\mathscr{H}(x)} \cong \tilde{k}$.
Type 4: Exercise.

Let K/k be a finite extension of non-archimedean valued fields. Let

$$e = [v(K^\times) : v(k^\times)]$$

be the ramification index and

$$f = [\tilde{K} : \tilde{k}]$$

the residue degree. Then

$$ef \leqslant [K : k]$$

with equality if k is discretely valued.

There exists an analogue for transcendental extensions:

Let K/k be a finitely generated extension of non-archimedean valued fields. Let

$$s = dim_{\mathbb{Q}} \left(\frac{v(K^\times)}{v(k^\times)} \otimes_{\mathbb{Z}} \mathbb{Q} \right)$$

and

$$t = \operatorname{tr.deg}(\tilde{K}/\tilde{k}).$$

Then

$$s + t \leqslant \operatorname{tr.deg}(K/k).$$

Proof. Choose $x_1, \ldots, x_s \in K^\times$ such that $[v(x_i)] \otimes 1$ are linearly independent over $\mathbb{Q}$. Choose $y_1, \ldots, y_t \in \mathcal{O}_K$ such that $\tilde{y}_1, \ldots, \tilde{y}_t$ are algebraically independent over $\tilde{k}$. We will prove that $x_1, \ldots, x_s, y_1, \ldots, y_t$ are algebraically independent over k. Assume there exists non-zero $p \in k[\underline{X}, \underline{Y}]$ such that $p(\underline{x}, \underline{y}) = 0$. Write

$$p(\underline{x}, \underline{y}) = \sum a_{\underline{i}\underline{j}} \underline{x}^{\underline{i}} \underline{y}^{\underline{j}}.$$

Choose $(\underline{i}_0, \underline{j}_0)$ such that $v(a_{\underline{i}\underline{j}} \underline{x}^{\underline{i}} \underline{y}^{\underline{j}})$ is minimal. We claim that every other monomial with minimal valuation has $\underline{i} = \underline{i}_0$. Note that $v(y_j) = 0$ for all $1 \leqslant j \leqslant t$.
Assume then that $v(a_{\underline{i}_0\underline{j}_0} x_1^{i_{01}} \cdots x_s^{i_{0t}}) = v(a_{\underline{i}_1\underline{j}_1} x_1^{i_{11}} \cdots x_s^{i_{1t}})$. We have

$$\sum_{i=1}^{t} (i_{0i} - i_{1i}) v(x_i) = v(a_{\underline{i}_1\underline{j}_1}) - v(a_{\underline{i}_0\underline{j}_0}) \in v(k^\times).$$

But the $v(x_i)$ lie in distinct cosets of $v(K^\times)/v(k^\times)$, and so $\underline{i}_0 = \underline{i}_1$. We divide $p(\underline{x}, \underline{y})$ by $a_{\underline{i}_0\underline{j}_0} x_1^{i_{01}} \cdots x_s^{i_{0t}}$ to get an expression

$$\sum b_{\underline{i}\underline{j}} \underline{y}^{\underline{j}} + C,$$

where $v(b_{\underline{i}\underline{j}}) = 0$ and $v(C) > 0$. Passing to the residue field, we get an algebraic dependence relation between the $\tilde{y}_j$: this is a contradiction.

We give one more classification of the points of $\mathbb{A}_k^{1,\mathrm{an}}$:

For $x \in \mathbb{A}_k^{1,\mathrm{an}}$, let $s(x) = dim_{\mathbb{Q}} \left(\frac{v(K(x))}{v(k)} \otimes_{\mathbb{Z}} \mathbb{Q} \right)$ and $t(x) = \mathrm{tr.deg}(\widetilde{K(x)}/\tilde{k})$. Then

Type	s	t	$\mathrm{tr.deg}(K(x)/k)$
1	0	0	0
2	0	1	1
3	1	0	1
4	0	0	1

We can define analytic functions on $\mathbb{A}_k^{1,\mathrm{an}}$:

Let U be an open subset of $\mathbb{A}_k^{1,\mathrm{an}}$. An *analytic function* on U is a map

$$F : U \to \coprod_{x \in U} \mathscr{H}(x)$$

such that for all $x \in U$ the following hold:
(1) $F(x) \in \mathscr{H}(x)$, and
(2) there exists a neighbourhood V of x and sequences $\{P_n\}$ and $\{Q_n\}$ of elements of $k[T]$ such that the Q_n don't vanish on V and

$$\lim_{n \to \infty} \sup_{y \in V} \left| F(y) - \frac{P_n(y)}{Q_n(y)} \right| = 0.$$

Setting $\mathcal{O}(U) = \{\text{analytic functions on } U\}$ makes $\mathbb{A}_k^{1,\mathrm{an}}$ into a locally ringed space.

$\mathcal{O}(\mathbb{A}_k^{1,\mathrm{an}})$ consists of power series with infinite radius of convergence. The local ring $\mathcal{O}_{|\ |_0}$ consists of power series with positive radius of convergence.

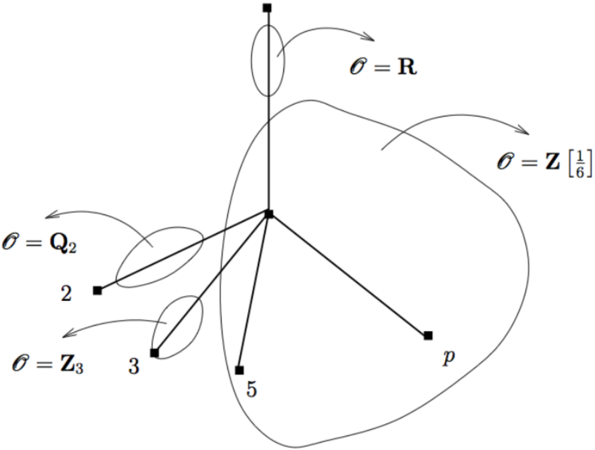


Figure: The structure sheaf of $\mathscr{M}(\mathbb{Z})$

Let $U = \mathscr{M}(\mathbb{Z}) \setminus |\ |_0$ and let $j : U \to \mathscr{M}(\mathbb{Z})$ be the natural inclusion map. Then $(j_* \mathcal{O}_U)_{|\ |_0}$ is the ring of adeles of $\mathbb{Q}$ and $(j_* \mathcal{O}_U^\times)_{|\ |_0}$ the group of ideles. This still works for $\mathbb{Z}$ replaced by any ring of integers.

Let $\mathscr{A}$ be a finite type k-algebra, e.g. $\mathscr{A} = k[T]$. For $f \in \mathscr{A}$, set $D(f) = \{x \in \mathscr{M}(\mathscr{A}) \mid f(x) \neq 0\}$. The map $\mathscr{M}(\mathscr{A}[1/f]) \to \mathscr{M}(\mathscr{A})$ induced by localisation is a homeomorphism onto its image $D(f)$.

This lets us identify $\mathbb{A}_k^{1,\mathrm{an}} \backslash | \ |_0 = D(T)$ with $\mathscr{M}(k[T, 1/T])$. Take another copy of $\mathbb{A}_k^{1,\mathrm{an}} = \mathscr{M}(k[T'])$ with $D(T') = \mathscr{M}(k[T', 1/T'])$. We can glue both copies of $\mathbb{A}_k^{1,\mathrm{an}}$ along $D(T)$ and $D(T')$ by the isomorphism induced by $k[T', 1/T'] \to k[T, 1/T] : T' \mapsto 1/T$. This gives the Berkovich projective line $\mathbb{P}_k^{1,an}$. It is Hausdorff, compact, and uniquely path-connected.

Note: there is also a Proj-type construction of $\mathbb{P}_k^{1,an}$.

Finally, let us prove that every entire non-constant function $f : \mathbb{A}_k^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ is surjective:

We know that $f = \sum_{i=0}^{\infty} a_i T^i$ is a power series with infinite radius of convergence. Subtracting a constant from f, it is enough to show that f has a root. We consider the Newton polygon $\mathcal{N}(f)$ of f: if $\mathcal{N}(f)$ has a segment of finite length n, then f has at least n roots. This can only fail if $\mathcal{N}(f)$ doesn't exist, i.e. if the points $(j, v(a_j))$ don't have a lower convex hull in $\mathbb{R}^2$. But f has infinite radius of convergence, so in particular $v_p(a_i/p^{ni}) \to \infty$ as $i \to \infty$ for all n, so the Newton polygon $\mathcal{N}(f)$ exists and f has a root.